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MODELLING OF DYNAMIC MEASUREMENTS FOR UNCERTAINTY ANALYSIS BY MEANS OF DISCRETIZED STATE-SPACE FORMS

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Abstract − Both, the ISO-GUM [1] and the Supplement S1 of the GUM [2] require expressing the knowledge about the measurement process by a so-called measurement function [3], which represents the mathematical relationship between the relevant parameters, the influence quantities, and the measurand(s). Nevertheless, both documents are confined to lumped-parameter systems in the steady state. Since dynamic measuring systems gain more and more importance, modern uncertainty determination must develop appropriate modelling approaches for dealing with dynamic measurements. This paper exemplarily describes a possible modelling approach for dynamic measurements that utilizes discretized state-space forms. The basic role of the cause-effect approach and its necessary inversion for the uncertainty evaluation is emphasized. The paper is an extension and refinement of former work of the authors [4].

Keywords: Dynamic measurements, state-space modelling

1. INTRODUCTION

For evaluating the measurement uncertainty, the GUM framework [1, 2] requires to express the knowledge about a measurement by the so-called measurement function [3], which represents the mathematical relationship between the relevant parameters, the influence quantities, the indication(s), and the measurand(s). But the GUM framework [1, 2] does not (yet) provide any assistance on modelling of measurements. Moreover, today it is confined to lumped-parameter systems in the steady state.

This paper describes a modelling approach that starts from a cause-effect analysis of the measurement process. For modelling of dynamic measurements in the time domain, discretized state-space forms are proposed. These mathematical forms originate from signal and system theory. Due to their obvious advantages (see Section 5), they form an appropriate means for modelling of measuring systems.

2. THE CAUSE-EFFECT APPROACH AND THE MEASUREMENT FUNCTION

In measurement, usually the measurand and other influence quantities can be seen as causative signals which are physically transformed by the measuring system into effects, for example into indications. Therewith, the measuring system assigns values to the measurand(s), and the system is influenced by system-disturbing influence quantities. The cause-effect approach is the most commonly used and comprehensible methodology for representing basic relationships in modelling of measurements [5, 6]. It is based on the constitutions of the path of the measurement signal from cause to effect. A model that describes the cause-effect behaviour of a measuring system or sensor is often termed 'measurement equation' or 'sensor equation'.

In contrast to this, for determining the measurement uncertainty, usually an 'inverse model' is needed that establishes the relationship between the 'target quantity', i.e. the measurand(s), $\boldsymbol{Y}$, and all relevant influence quantities and the indication(s), $\boldsymbol{X} = (X_1,..., X_N)^T$. So far, this model category has been termed 'model equation' or 'measurement reconstruction model' [5].

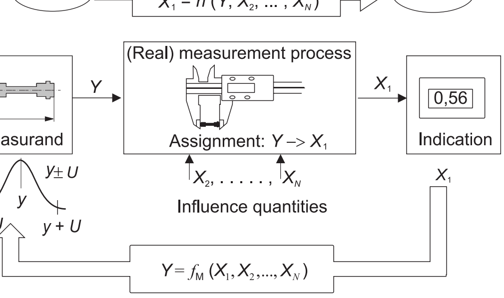

Fig. 1. Comparison of model categories: 'Measurement equation' vs. 'model equation' or 'reconstruction model' or 'measurement function' [4-6].

The new ISO IEC Guide 99 ('VIM 3') [4] uses the term 'measurement function' which is generally expressed as

$$\boldsymbol{Y} = f_M (X_1,..., X_N). \qquad (1)$$

Fig. 1 illustrates the difference between the two model categories.

In practice, due to its comprehensibility and deducibility from the real system, the cause-effect approach almost always forms the basis for the modelling of measurements. The cause-effect approach itself is founded on the transfer behaviour of the functional elements of the measuring chain.

3. DESCRIBING AND MODELLING THE TRANSFER BEHAVIOUR

Measuring systems are usually modelled the same way as any other technical information system. First, the system is decomposed and modularized into functional elements. Then, the transmission behaviour of each functional element is mathematically described [5-6]. The so-called transfer function [7] relates the output signal(s) to the input signal(s):

$$\boldsymbol{X}_{OUT} = h\, (\boldsymbol{X}_{IN}) , \qquad (2)$$

where $\boldsymbol{X}_{IN} = (X_{IN1},..., X_{INn})^T$ – input signal(s), $\boldsymbol{X}_{OUT}$ – output signals, and h – transfer function.

Fig. 2 exemplarily shows a depiction of a general steady-state transmission element (a) and its application to an example (b) [6].

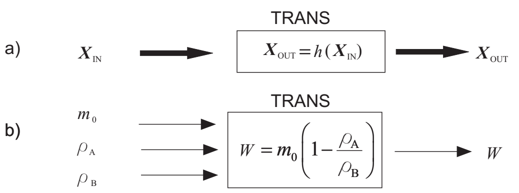

Fig. 2. General transmission element: (a): General depiction. (b): Example: air buoyancy correction. Symbols: h – transfer function; $(X_{IN1},..., X_{INn})^T = \boldsymbol{X}_{IN}$ – input signal(s); $\boldsymbol{X}_{OUT}$ – output signals; W – air buoyancy correction in terms of mass; ρ_A – air density, ρ_B – density of the body; m_0 – uncorrected (true) mass [6].

In measurement, the great majority of systems are treated as being linear and time-invariant [6]. Therefore, a proper description of this system category is of great importance in metrology and industrial measurement. Moreover, today, in analytical metrology, it is best practice to treat even slightly nonlinear and time-variant systems this way with accountting for additional uncertainty contributions owing to nonlinearity and dynamic effects [6].

The transfer function of a time-invariant system or transmission element is represented by an algebraic equation (see Table 1). For a linear system, the transfer function consists of constant transmission factors,

$$\boldsymbol{X}_{OUT} = h\, (\boldsymbol{X}_{IN}) = \boldsymbol{A} = (A_1,..., A_m)^T , \qquad (3)$$

where $\boldsymbol{A} = (A_1,..., A_m)^T$ are constant factors.

Linear and time-invariant transfer functions can easily be inverted into the so-called ‘measurement function’ [3] or 'reconstruction model' (see Equation (1) and Section 2).

But to an increasing extend, dynamic measuring systems gain importance in metrology and industrial measurement. The time-dependent behaviour of these systems or transmission elements results from transient and storage effects affecting the quantity of interest. This might be briefly

System class			Equation for general systems	Equation for linear systems
Static systems		Algebraic equation	Equation without time dependencies: $Y = f(X)$	Linear system of equations: $Y = AX$
Dynamic systems	Description in time domain	Differential equation	$\underline{Y}(t) = f\left(Y^{(1)}, Y^{(2)},..., X, X^{(1)}, X^{(2)},...,t\right)$	Linear differential equation: $\sum_{\upsilon=1}^{n} a_\upsilon \underline{Y}^{(\upsilon)} = \sum_{\mu=1}^{m} b_\mu \underline{X}^{(\mu)}$
		State space model	$Z(t) = f\left(Z(t), X(t), t\right)$ $Y(t) = g\left(Z(t), X(t), t\right)$	$\dot{Z}(t) = AZ(t) + BX(t)$ $Y(t) = CZ(t) + DX(t)$
	Description in frequency domain	Transfer function		$Y(s) = G(s)\underline{X}(s)$ $G(s) = \frac{\sum_{\mu=0}^{m} b_\mu s^\mu}{\sum_{\upsilon=0}^{n} a_\upsilon s^\upsilon}$

Table 1. Survey on static and dynamic systems along with tools for their appropriate mathematical description.

explained with an example [4, 6, 10]: A liquid-in-glass thermometer that indicates the ambient air temperature ϑ_a plus its (statical) instrumental error, $\vartheta_{IND} = \vartheta_a + \Delta\vartheta_{INSTR}$, is at the time t_0 being immersed into a water bath with temperature ϑ_B. Then, the cause-effect relationship of the measurement and temperature equalization process may by expressed by the following differential equations

$$\vartheta_{IND} = \vartheta_B + \Delta\vartheta_{INSTR} - T\,\frac{d\vartheta_{Th}}{dt} \;. \qquad (4)$$

Consequently, the model equation becomes

$$\vartheta_B = \vartheta_{IND} + \Delta\vartheta_{INSTR} - T\,\frac{d\vartheta_{Th}}{dt} \;, \qquad (5)$$

where $T \cdot \frac{d\vartheta_{Th}}{dt} = \delta\vartheta_{DYN}\,(t)$ can be seen as dynamic error component, whose expectation is approximately

$\delta\vartheta_{DYN}\,(t) = (\vartheta_B - \vartheta_a) \cdot \exp\left(\frac{t - t_o}{T}\right)$ [5-6, 9].

In general, dynamic measuring systems can be classified as lumped-parameter systems or distributed-parameter systems. The key characteristic of a lumped-parameter system is that the state of the system, which uniquely describes the system behaviour, depends only on time. In the time domain, it is generally described by a set of ordinary differential equations [7]. Table 1 gives an overview on the mathematical tools used for the description of analogue static and dynamic systems in both the time domain and the frequency domain [6-8].

It should be emphasized that in today's practice, the system description is usually discretized. Discretization allows for treating many types of systems as being linear (at a discrete point of time) and offers advantages for digital signal processing [11].

4. INVERTING THE TRANSFER FUNCTION

Whereas for linear and linearizable systems, the measurement function [3] is usually established by algebraically inverting the mathematical cause-effect relationship expressed by the transfer function, in case of non-linearizable and dynamic measuring systems this might be awkward, i.e., in case of so-called ill-posed inverse problems.

Alternatively to algebraically inverting the cause-effect relationship, for uncertainty evaluation, the following strategies might be applied:

(a) Incorporating the mathematical cause-effect-relationship as a so-called ‘Model Prior’ into the ‘Likelihood’ of the Bayes Theorem [9] and computing the ‘Joint Posterior’ probability density function (pdf) for the measurand.

(b) Estimating the parameters of the measurement function [3] by means of recursive estimation algorithms (see Section 4), such as, for example, Kalman Filters.

(c) Combinations of (a) and (b).

5. STATE-SPACE FORMS

State-space forms are a useful alternative approach to describing dynamic measurements in the time domain. In general, they consists of a combination of a system equation (6) and a so-called output equation (7) [8, 9] according to

$$\dot{\boldsymbol{Z}} = f_{\mathrm{S}}\left[\boldsymbol{Z}(t),\ \boldsymbol{X}_{\mathrm{IN}}(t),\ t\right], \tag{6}$$

$$\boldsymbol{X}_{\mathrm{OUT}}(t) = f_{\mathrm{OUT}}\left[\boldsymbol{Z}(t), \boldsymbol{X}_{\mathrm{IN}}(t)\right], \tag{7}$$

where the state vector $\boldsymbol{Z}$ represents the present state of the system. For example, an appropriate state variable (vector) may be the (real) temperature of a thermometer immersed into a water bath (see Section 3).

State-space forms are mathematically equivalent to the description by means of ordinary differential equations (see (4)). The relevant advantages are:

(a) technically easy interpretation of the state vector

(b) having first order differential equations only

(a) allowing to easily derive the input and the output quantities/vectors from the state vector/variable.

Additionally, time discretization results in a finite-state form that basically allows to treat a measuring system as being linear (and time-invariant) at a discrete state $\boldsymbol{Z}_k$ [11]. Consequently, (6) and (7) become

$$\boldsymbol{Z}_{k+1} = \boldsymbol{A}_k\, \boldsymbol{Z}_k + \boldsymbol{B}_k \cdot \boldsymbol{X}_{\mathrm{IN}k}, \tag{8}$$

$$\boldsymbol{X}_{\mathrm{OUT}k} = \boldsymbol{C}_k\, \boldsymbol{Z}_k\ (+\ \boldsymbol{D}_k\, \boldsymbol{X}_{\mathrm{IN}k}), \tag{9}$$

where k indicates a discrete point in time and $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$ and $\boldsymbol{D}_k$ representing constant transmission vectors at k.

With a view to evaluate the (measurement) uncertainty for a (measurement) process described in space-state form, the above variables (vectors) $\boldsymbol{X}_{\mathrm{IN}}$, $\boldsymbol{X}_{\mathrm{OUT}}$ and $\boldsymbol{Z}$ are to be described by appropriate probability density functions (pdf) $g(\boldsymbol{X}_{\mathbf{IN}k})$, $g(\boldsymbol{X}_{\mathrm{OUT}k})$ and $g(\boldsymbol{Z}_k)$, which represent the incomplete knowledge about the variables (vectors). Furthermore, based on the existing knowledge about the measuring system, both the state equation and the output equation may be augmented by additional noise/uncertainty components to account for the imperfection in modelling of the whole (measurement) process.

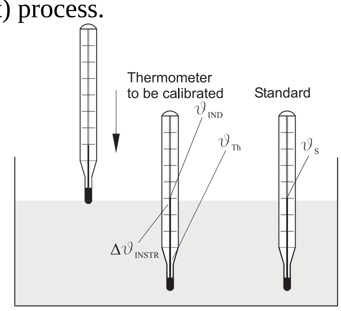

Fig. 3. Illustration of the example described: Modelling of dynamic error and uncertainty.

For better illustration of the application of state-space forms to measuring systems, the thermometer example given in Section 2 is changed and extended to a calibration of the instantly immersed thermometer, and the bath temperature is made known by a standard thermometer (see Fig. 3) [4, 10]. Assume, the calibration aims at the (steady-state) systematic error $\Delta\vartheta_{\mathrm{INSTR}}$. Obviously, the (real) temperature of the thermometer to be calibrated, ϑ_{Th}, might be taken as a state variable, and the bias-corrected temperature indicated by the standard, ϑ_{s}, is an appropriate system input. Then, the discretized system and output equations would formally read as

$$\boldsymbol{Z}_{k+1} = \boldsymbol{A}_k \cdot \boldsymbol{Z}_k + \boldsymbol{B}_k \cdot \Delta\vartheta_{\mathrm{INSTR}} + \boldsymbol{B}_k \cdot \vartheta_{\mathrm{s}}. \tag{10}$$

$$\boldsymbol{X}_{\mathrm{OUT}k} = \vartheta_{\mathrm{IND}k} = \boldsymbol{Z}_k + \boldsymbol{\nu}_k\,, \tag{11}$$

where $\boldsymbol{\nu}_k$ represents a random uncertainty component. Fig. 4 illustrates the basic structure of this model [4, 10].

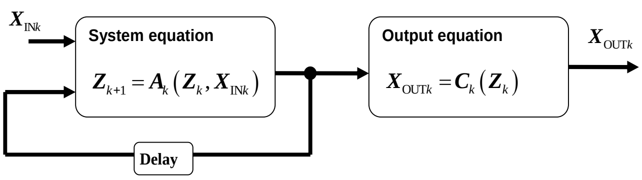

Fig. 4. Basic structure of a discretized state-space model in accordance with Equations (10) and (11) [4, 11].

6. MODEL-BASED ESTIMATOR

Assume the above example (see Figure 4 and Equations (10) and (11) in Section 3): At the end of a thermometer-production process, the instruments are calibrated, and the measurand is the (steady-state) instrumental error. The calibration is carried out by immersing the thermometers into a water bath of known temperature. For efficiency reasons, one cannot wait until the thermal steady state is reached. Therefore, a good estimate of the unknown dynamic error is needed. This estimation can be carried out on the basis of a state-space model [4, 7, 10]. Fig. 5 illustrates the idea [10]: Both the uncertainties for the system equation and the output equation are taken into consideration. The system input and the state vector are described by appropriate PDFs. Due to the fact that in the given example the output quantity $X_{\mathrm{OUT}k}$, which is chosen to be the indication of the instrument to be calibrated (see Equations (10 and (11)), is well known, the (easy obtainable) inverse output equation might be used for obtaining a second estimate of the state-vector PDF g_{L} ($\boldsymbol{Z}_k$) that is derived from real measurement data. Employing the *Bayes* theorem, this estimate is used to permanently update the PDF g_{p} ($\boldsymbol{Z}_k$) provided by the system equation. For an optimal estimation result, g_{e} ($\boldsymbol{Z}_k$), possible systematic uncertainty contribution, which can result in a significant covariance of the states $\boldsymbol{Z}_k$ and $\boldsymbol{Z}_{k+1}$, are to be taken into consideration. Therefore, the estimation algorithm used for the 'Bayesian step' (see Figure 5) must be capable to cope with unknown correlation, by employing, for example, so-called covariance bounds [10, 12].

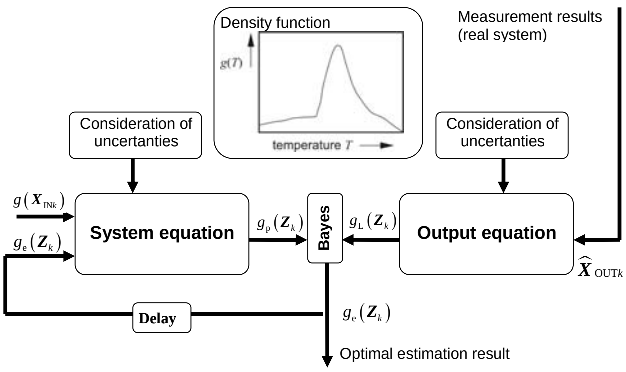

Fig. 5. Model-based estimator [4,10] for the example given (see [4]).

Based on real input data, this model-based estimator has successfully been proven [4, 10].

The possible physical definitions and allocations of the state-space vectors (see Equations (8) and (9)) to a particular measurement process mainly depends on the model structure of the process or system and, hence, on the measurement method [3] utilized [5].

7. CONCLUSION

Modelling the measuring process is a necessary task for evaluating measurement results and ensuring their reliability. Since dynamic measurements gain more and more importance, modern uncertainty evaluation must develop appropriate modelling approaches. It is exemplarily demonstrated that discretized state-space forms in connection with model-based estimators are a suitable alternative for modelling dynamic measurements in uncertainty evaluation. First results show the performance and the potential of this approach.

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