CUBIC SPLINE VIRTUAL INSTRUMENTS ASSOCIATED TO SYMBOLIC TRANSMITTANCE OF CIRCUIT

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Abstract- This paper develops the synthesis of some elementary cubic spline virtual instruments associated to the symbolic transmittance of circuits. The implementation of these virtual instruments is based on the power of LabVIEW programming language to develop modular applications. The method has a pronounced educational purpose and can be used in electric and electronic circuits time analysis. The general symbolic transmittance of such circuits is expressed like a linear combination of elementary symbolic transmittances. An elementary first or second order differential equation is associated to an elementary symbolic transmittance. The solutions of this equation are given by a third order spline approximates and the algorithms lead to the synthesis of inferior order spline virtual instruments. Using a cascade connection of such elementary virtual instruments, a high order virtual instrument associated to the general symbolic transmittance is obtained. Finally, the circuit response can be determined.

I Introduction. Symbolic Transmittance [1]

Circuit analysis represents an essential part of the teaching process dedicated to the electronic and electrical engineering students. The numerical methods are widely used for this analysis and they are based on complex mathematical calculus.

For a common circuit, the n order differential equation, with constant coefficients, looks like this [1]:

$$\begin{aligned} a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + a_{n-2} y^{(n-2)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = \\ = b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \ldots + b_1 x'(t) + b_0 x(t) \end{aligned} \tag{1}$$

We introduce the derivable operator "p" for an x(t) function as [1]:

$$\frac{dx(t)}{dt} = px(t) \tag{2}$$

The derivable operator p=d/dt can be considered as an algebraic variable for which the addition, subtraction and multiplication operations remain valid. It is important not to confuse this operator with variable "s" of Laplace transformations, which is a complex variable. By applying the algebraic operator "p" to the equation (1), we obtain the symbolic relation:

$$[a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0] \cdot y(t) = [b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0] \cdot x(t) \tag{3}$$

It results that [1]:

$$y(t) = \frac{b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0} \cdot x(t) \qquad \text{or} \qquad y(t) = \Im(p) x(t) \tag{4}$$

where $\Im(p)$ is called the global symbolic transmittance of the circuit [1].

By decomposing the global symbolic transmittance in partial symbolic transmittances, we will write for most of the applications:

$$\Im(p) = \prod \Im_{mk,nk}(p) \tag{5}$$

where $\Im_{mk.nk}(p)$ is a ratio of first or second order polynomials and will be called symbolic elementary transmittances; $mk \le nk \le 2$

There are the following types of elementary symbolic transmittances:

- elementary symbolic transmittances of second order:

- $$\Im_{22}(p)=\frac{\sum_{k=0}^{2} b_k p^k}{\sum_{k=0}^{2} a_k p^k} \qquad \Im_{12}(p)=\Im_{22}(p)\Big|_{b_2=0} \qquad \Im_{02}(p)=\Im_{12}(p)\Big|_{b_1=0} \qquad (6)$$
- reference symbolic transmittance of second order:

$$\Im_{02}^{0}(p)=\Im_{02}(p)\Big|_{b_0=1} \qquad (7)$$

- elementary symbolic transmittances of first order :

- $$\Im_{11}(p)=\frac{\sum_{k=0}^{1} b_k p^k}{\sum_{k=0}^{1} a_k p^k} \qquad \Im_{01}(p)=\Im_{11}(p)\Big|_{b_1=0} \qquad (8)$$
- - reference symbolic transmittance of first order:

$$\Im_{01}^{0}(p)=\Im_{01}(p)\Big|_{b_0=1} \qquad (9)$$

II. Spline Virtual Instruments Associated To Symbolic Transmittances Of Circuits

Using Labview programming software, which provides a very good input data control, a high flexibility in data analysis and the possibility to develop modular applications, the symbolic transmittances $\Im_{02}(p), \Im_{12}(p), \Im_{22}(p)$ will be implemented using the second order **T_{02}[], T_{12}[], T_{22}[]** virtual instruments [6, 8]. Similarly the symbolic transmittances $\Im_{01}(p)$ and $\Im_{11}(p)$ will also be created using the first order **T_{01}[ ], T_{11}[ ]** virtual instruments.

A. The T_{02}[]T_{12}[ ] And T_{22}[ ] Virtual Instruments

For the **T_{02}[]** virtual instrument we consider the second order symbolic transmittance with the next associated differential equation [1,7]:

$$F_2[a_2 \cdot y^{(2)}]=F_0[b_0 x] \qquad (10)$$

or explicitly :

$$a_2 y''+a_1 y'(t)+a_0 y(t)=b_0 \cdot x(t) \qquad (11)$$

The relation (11) can be written as follows:

$$y''=A\cdot x(t)+B\cdot y+C\cdot y' \qquad (12)$$

where:

$$A=\frac{b_0}{a_2},\quad B=-\frac{a_0}{a_2},\quad C=-\frac{a_1}{a_2} \qquad (13)$$

We will create a spline virtual instrument **T_{02}[]** based on the spline approximate solution of the differential equation (26). Spline cubical polynomial functions are used in the algorithm [10, 11, 12]. We will consider a uniform division of the input signal's time support, $[0,t_N]$, that has an iteration step of $h=t_{i+1}-t_i$:

$$\Delta := t_0 < t_1 < ... < t_i < t_{i+1} < ... < t_N; \quad t_0=0 \qquad (14)$$

An elementary cubical spline function is defined on $[t_i, t_{i+1}]$:

$$q_i(t)=q_{i-1}(t_i)+q'_{i-1}(t_i)(t-t_i)+\frac{1}{2}q''_{i-1}(t_i)(t-t_i)^2+\frac{1}{6}\alpha_i(t-t_i)^3 \qquad (15)$$

For $t_i=t_0$

$$q_0(t)=q_0+q'_0(t-t_0)+\frac{q''_0}{2}(t-t_0)^2+\frac{\alpha_0}{6}(t-t_0)^3 \qquad (16)$$

For $t=t_1$; $h=t_1-t_0$

$$q_0(t_1)=q_0+q'_0 h+\frac{q''_0}{2}h^2+\frac{\alpha_0}{6}h^3$$

$$q'_0(t_1)=q'_0+q''_0 h+\frac{\alpha_0}{2}h^2 \qquad (17)$$

$$q''_0(t_1)=q''_0+\alpha_0 h$$

But: $$q''_0(t_1)=F_2\left[x(t_1),q_o(t_1),q'_0(t_1)\right] \qquad (18)$$

By replacing, we obtain:

$$q_0'' + \alpha_0 h = Ax(t_1) + B\left[q_0 + q_0 h + \frac{q_0''}{2}h^2 + \frac{\alpha_0}{b}h^3\right] + C\left[q_0' + q_0'' h + \frac{\alpha_0}{2}h^2\right] \quad (19)$$

By writing:

$$D = h - B\frac{h^3}{6} - C\frac{h^2}{2} \quad (20)$$

We obtain :

$$\alpha_0 = \frac{1}{D}\left\{-q_0'' + Ax(t_1) + B\left[q_0 + q_0' h + \frac{1}{2}y_0'' h^2\right] + C\left[q_0' + q'' h\right]\right\} \quad (21)$$

For $t = t_{i+1}$

$$\begin{aligned} q_i(t_{i+1}) &= q_{i-1}(t_i) + q_{i-1}'(t_i)h + \frac{1}{2}q_{i-1}''(t_i)h^2 + \frac{\alpha_i}{6}h^3 \\ q_i'(t_{i+1}) &= q_{i-1}'(t_i) + q_{i-1}''(t_i)h + \frac{\alpha_i}{2}h^2 \\ q_i''(t_{i+1}) &= q_i''(t_i) + \alpha_i h \end{aligned} \quad (22)$$

With condition:

$$q_i''(t_{i+1}) = F_2\left[x(t_{i+1}), q_i(t_{i+1}), q_i'(t_i)\right] \quad (23)$$

By replacing:

$$q_{i-1}''(t_i) + \alpha_i h = Ax(t_{i+1}) + B\left[q_{i-1}'(t_i) + q_{i-1}'(t_i)h + \frac{1}{2}q_{i-1}''(t_i)h^2 + \right] + C\left[q_{i-1}'(t_i) + q_{i-1}''(t_i)h + \frac{\alpha_i}{2}h^2\right] \quad (24)$$

The coefficient α_i is obtained:

$$\alpha_i = \frac{1}{D}\left\{-q_{i-1}''(t_i) + Ax(t_{i+1}) + B\left[q_{i-1}(t_i) + q_{i-1}''(t_i)h + \frac{1}{2}q_{i-1}(t_i)h^2\right] + C\left[q_{i-1}'(t_i) + q_{i-1}''(t_i)h\right]\right\} \quad (25)$$

The convergence and the uniqueness of the solution are analysed in papers [1, 3, and 7].
The approximate solution is obtained using the **T02[]** virtual instrument, which implements the previous relations. The block diagram and the icon of this instrument are presented in Figure 1.

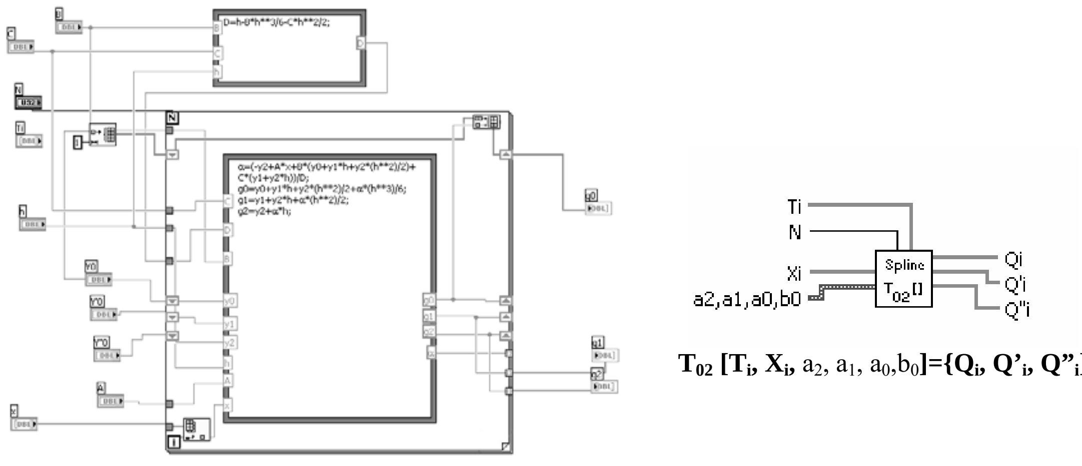

Figure 1 The block diagram and the icon of **T02[]** virtual instrument

The significance of the input and output values are: $\mathbf{T_i}=\{t_i\}$; $t_i \in [0,\ t_N]$; $\mathbf{X_i}=x\{t_i\}$ $\mathbf{Q_i}=\{q_i(t_{i+1})\}$; $\mathbf{Q'_i}=\{q'_i(t_{i+1})\}$; $\mathbf{Q''_i}=\{q''_i(t_{i+1})\}$; a_2, a_1, a_0, b_0 - are set according to the circuit type; N- the number of points; h - the length of $[t_{i-1},\ t_i]$ interval. The output vectors of the **T02[ ]** virtual instrument are the approximate response, respectively the first and second approximate response derivatives.

$$\mathbf{Y_i \approx Q_i;\ Y'_i \approx Q'_i;\ Y''_i \approx Q''_i;} \quad (26)$$

In the **T02[ ]** structure, by making $b_0 = 1$, we obtain the reference $\mathbf{T^0_{02}[\ ]}$ virtual instrument:

$$\mathbf{T^0_{02}[\ ] = T_{02}[\ ]}\ \Big|_{b0=1} \quad (27)$$

The symbolic transmittance **T12[ ]** is described by the relation (17). By expanding we can write:

$$\Im_{12}(p) = \Im_{02}^0(p)\cdot(b_0 + b_1 p) = \Im_{02}^0(p)\cdot\Im_{10}(p) \quad (28)$$

We may notice that the **T₁₂[ ]** virtual instrument associated to the symbolic transmittance $\Im_{02}^{0}(p)$ supplies the **Q_i, Q'_i, Q''_i** outputs. For the complete time support $[0,t_N]$ the circuit's response is given by:

$$\mathbf{Y_i} \approx b_0 \cdot \mathbf{Q_i} + b_1 \cdot \mathbf{Q'_i} \qquad (29)$$

The **T_{12}[ ]** virtual instrument synthesis will have the structure from Figure 2.

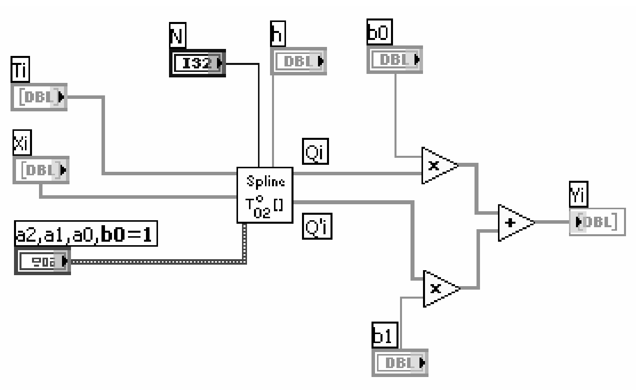

Figure 2 – Synthesis of the **T_{12}[ ]** virtual instrument

Relation (6) describes the symbolic transmittance $\Im_{22}(p)$. Similarly the synthesis of **T_{22}[ ]** virtual instrument associated to symbolic transmittance $\Im_{22}(p)$ is based on the circuit's answer vector:

$$\mathbf{Y_i} \approx b_0 \cdot \mathbf{Q_i} + b_i \cdot \mathbf{Q'_i} + b_2 \cdot \mathbf{Q''_i} \qquad (30)$$

B. The T_{01}[ ] And T_{11}[ ] Virtual Instruments

A **T_{01}[ ]** spline virtual instrument will be created using the approximate solution of the next equation:

$$a_1 y'(t) + a_0 y(t) = b_0 \cdot x(t) \qquad (31)$$

Due to the weak convergence of the approximate solution, we use an improved iterative algorithm that calculates the solution in the middle of each interval $[t_i, t_{i+1}]$, where h=. $t_{i+1}-t_i$. We consider the circuit equation:

$$y'(t) = F_1[x(t), y(t)] = Ex(t) + Fy(t) \qquad (32)$$

with initial conditions :

$$y(t_0) = y_0,\ y'(t_0) = F_1[x(t_0), y_0] \qquad (33)$$

According to relation (1) we have n = 1 and we choose m = 3, so that we build cubic polynomial Spline functions. After simple operations it results:

$$q_0 = q_0 + 2 \cdot q'_0 \cdot \left(\frac{h}{2}\right) + \frac{5}{2} \cdot q''_0 \cdot \left(\frac{h}{2}\right)^2 + \frac{7}{3} \cdot a_0 \cdot \left(\frac{h}{2}\right)^3$$

$$q'_0 = q'_0 + 2 \cdot q''_0 \cdot \left(\frac{h}{2}\right) + \frac{5}{2} \cdot a_0 \cdot \left(\frac{h}{2}\right)^2 \qquad (34)$$

$$q''_0 = q''_0 + 2 \cdot a_0 \cdot \left(\frac{h}{2}\right)$$

$$a_0 = \frac{1}{H}\left\{ Ex(t_1) + F\left[q_0 + y' \cdot 2 \cdot \left(\frac{h}{2}\right) + \frac{5}{2} \cdot q'' \cdot \left(\frac{h}{2}\right)^2 \right] - q'_0 - \frac{1}{2} q''_0 \cdot 4 \cdot \left(\frac{h}{2}\right) \right\} \qquad (35)$$

where

$$H = \frac{1}{4} \cdot \left(\frac{h}{2}\right)^2 - F \frac{7}{3} \left(\frac{h}{2}\right)^3 \qquad (36)$$

The approximate solution is obtained by using **T_{01}[ ]** virtual instrument with the structure described by the equation:

$$\mathbf{T_{01}[T_i, X_i, Q_i,}\ a_0,\ b_0\mathbf{] = \{Q_i, Q'_i\}} \qquad (37)$$

The structure of the **T_{01}[ ]** and **T_{02}[ ]** virtual instruments is similar. The difference consists in the mathematical algorithm, which includes a convergence correction.

In a similar way with **T_{22}[ ],** the synthesis **T_{11}[ ]** virtual instrument can be obtained.

C. Virtual Instrument Associated To Global Symbolic Transmittance T[ ],

A virtual instrument associated to the global symbolic transmittance T[ ] can be obtained by connecting the first and second order type virtual instruments, function of decomposing of the global symbolic transmittance $\Im(p)$ in elementary symbolic transmittances. For example the global symbolic transmittance:

$$\Im_{13}(p) = \frac{b_1 p + b_0}{a_2 p^2 + a_1 p + a_0} \cdot \frac{1}{a'_1 p + a'_0} \tag{38}$$

can be decomposed:

$$\Im_{13}(p) = \Im_{12}(p) \cdot \Im_{01}^{0}(p) \tag{39}$$

The next virtual instrument will result:

$$\mathbf{T_{13}[\ ] = T_{12}[\ ] \cdot T^{0}_{01}[\ ]} \tag{40}$$

The output vector $\mathbf{Q_i}$ of $\mathbf{T_{12}}$ will become the input vector for $\mathbf{T^{0}_{01}[\ ]}$ virtual instrument. The synthesis of $\mathbf{T_{13}[\ ]}$ virtual instrument is shown in Figure 3.

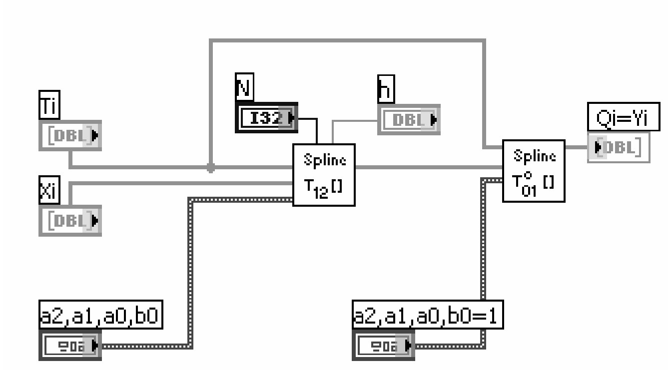

Figure 3.– Synthesis of $\mathbf{T_{13}[\ ]}$ virtual instrument; first synthesis possibility

III. Application

We consider a forth order low pass filter (LPF) with the transmittance given by the equation:

$$\Im_{04}(p) = \Im_{02}(p) \cdot \Im_{02}(p) = \frac{A_u \omega_r^2}{d_1 p^2 + c_1 \omega_r p + \omega_r^2} \cdot \frac{\omega_r^2}{d_2 p^2 + c_2 \omega_r p + \omega_r^2} \tag{41}$$

In relation (41) A_u is the gain; p is the derivate operator; ω_r is the reference frequency defined in a relation with a point situated in the transfer characteristic; c_1, d_1, c_2, d_2, are coefficients that depend on the transfer function type. (For our application these coefficients have the next values c_1= 2.1850, d_1= 5.5340, c_2= 0.1960, d_2= 1.2000 and they correspond to a Chebyshev forth order filter [2])
The coefficients of equation (12) will be:

$$A = \frac{\omega_r^2}{d_k}; \quad B = -\frac{\omega_r^2}{d_k}; \quad C = -\frac{a_k}{d_k}\omega_r \qquad (k = 1,2) \tag{42}$$

We consider a unit impulse vector $\mathbf{X_i=D_i=[}$1,0,0,…0$\mathbf{]}$ of N length applied to the input of the forth order LPF virtual instrument. The implementation of this application implies the cascade connection of two $\mathbf{T_{02}[\ ]}$ elementary virtual instruments. The, ω_r, a_1, b_1, a_2, b_2 (A_u=1) specific values permit to calculate, with a formula node structure, the input values A, B, C for the first and the second $\mathbf{T_{02}[\ ]}$ virtual instruments. The virtual instrument $\mathbf{T_{04}[\ ]}$ outputs the approximate response h[n]. By using an FFT operator, the Fourier transform of the h[n] signal and implicitly the frequency characteristics $|\text{FFT } h[n]|$ of Chebyshev forth order low pass filter described by the transmittance from relation (11) are calculated. The results are presented on the virtual instrument panel (Figure 4).

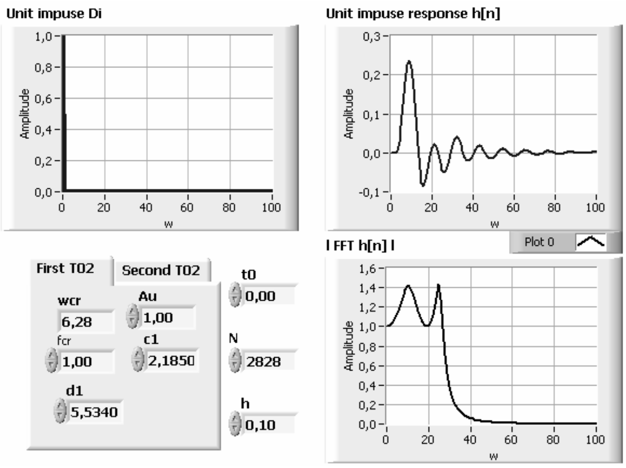

Figure 4 Chebyshev LPF forth order Spline virtual instrument-front panel.

IV. Conclusions

The symbolic solutions of differential equations associated to electric circuits lead to symbolic transmittances $\Im(p)$, expressed by the p derivation operator. Using the modular proprieties of LabVIEW graphical programming language a cubic spline virtual instrument **T[ ]** associated to the circuit global symbolic transmittance is created and provides the response for a certain input signal.
The Spline virtual instrument are conceived as an approximate analyse procedure. It is successfully used when the "p" inverse transformation cannot be exactly realised. The advantage of the method consists in the simplicity of the mathematical instrument that has been used.
The method can be successfully used as a teaching tool starting with the 2nd year of studies, when mathematical and circuits' theory knowledge is already known by students. Our virtual instruments allow the analysis of linear, concentrated and stationary circuits for any signal waveform and can be used in circuits' analysis, modelling and testing processes.
The method represents a teaching tool for basic theories and fundamentals of numerical methods; it helps students to acquire skills needed for implementing a computer solution; and finally it provides an environment where they can familiarize with LabVIEW software and its use in solving engineering problems.

V. References

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