Stability Analysis of Oscillators Based on a Delta–Sigma Topology

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Abstract – *The growing demand of mixed–signal integrated circuits encourages the research of Built–In Self–Test (BIST) techniques to achieve simpler and less expensive testing processes. High quality sinusoidal oscillators based on a Δ–Σ topology are an effective solution to perform the test of this kind of devices. Unfortunately, due to the in–loop 1–bit quantizer nonlinearity, several problems of stability have been observed. A stability analysis on the behavior of the oscillator based on a second order Δ–Σ modulator presented in [1] is described in this paper. In particular, it is shown that the oscillator is intrinsically unstable and its complete dynamics is very difficult to predict exactly. Finally, a possible stabilization strategy is proposed.*

Keywords – BIST, oscillator, delta–sigma.

I. Introduction

One of the most demanding problems dealt with by microelectronic industry, is the increase in testing costs associated with the production of Integrated Circuits (ICs). This has been caused by the growth in complexity of the electronic devices and by the simultaneous reduction in the costs of chip fabrication processes. This trend has been further stressed by the widespread diffusion of mixed–signal integrated circuits and by the competitive price pressures and severe time–to–market demands of consumer electronics. As a result, nowadays costs for testing mixed–signal circuits may cover nearly 50% of the overall production budget [2], [3].

Built–In Self–Test (BIST) techniques allow a drastic cut in such expenses through the integration of various testing resources directly on chip, thus increasing the controllability and observability of circuital parameters. An effective mixed–analog/digital BIST scheme results from the tradeoff between features such as reliability, high precision, programmability, flexibility, low cost of integration, and low analog complexity [4]. Since often analog–to–digital converters (ADCs) are the key components of mixed–signal circuits, it is useful to insert low cost, high quality sinusoidal waveform generators in the BIST schemes used to test these devices. Sinusoidal oscillators based on a Δ–Σ topology, i.e. digital resonators exploiting 1–bit delta–sigma properties, possess all of the above stated features. Particularly, they avoid the use of hardware multipliers and reduce the analog part of the mixed–signal circuit to the use of a 1–bit DAC only. Several implementations of Δ–Σ resonator–modulators have been proposed [1], [4]. In this paper it will be shown the critical behavior of one such implementations [1]. At first, some results about Δ–Σ resonator–modulators stability are illustrated. Then, a stability analysis of the oscillator based on the root locus technique is described. Finally, a possible solution is proposed to stabilize the circuit.

II. Stability issues in delta–sigma resonator–modulators

Most of Δ–Σ resonator–modulators derive from the harmonic digital resonator based on the cascade of two discrete–time integrators in a loop, with the sign of one integrator being positive and the other negative [1]. In these structures, the oscillation frequency depends on one or two multiplicative coefficients. In order to avoid the use of hardware multipliers and multi–bit D/A converters, a Δ–Σ modulator can take the place of one of the registers in the loop. The increase in circuital complexity due to the introduction of the modulator is largely counterbalanced by the benefit of using a single–bit output, some shift registers and a single–bit multiplier, i.e. a multiplexer. In particular, if a second order Δ–Σ modulator is employed, the circuit presents the structure shown in Fig. 1, where the frequency of the self–generated sinusoidal oscillation is given by the product $a_{12}a_{21}$, and the signal amplitude depends both on a_{12}, a_{21} and on the system initial conditions ($x_1[0]$, $x_2[0]$, $u_1[0]$, $v_1[0]$, $y[0]$) [1]. If the parameter a_{12} is chosen as a power of 2, the corresponding multiplier can be implemented with a shift register. Moreover, the multiplication by a_{21} can be achieved using $+a_{21}$ and $-a_{21}$ as the two inputs of a multiplexer, controlled by the 1–bit output $y[\cdot]$. By employing a white noise input to represent the behavior of the 1–bit quantizer, the Noise Transfer Function (NTF) of the resonator-modulator on the whole is given by:

$$NTF(z) = \frac{1 - 2z^{-1} + z^{-2}}{1 + (a_{12}a_{21} - 2)z^{-1} + z^{-2}} \cdot NTF'(z) \quad (1)$$

where $0 < a_{12}a_{21} < 4$ and $NTF'(z) = (1 - z^{-1})^2$ is the ordinary noise transfer function of the internal 2nd–order Δ–Σ modulator. It can be shown easily that a couple of poles and a couple of zeros are inserted on the unit circle by the cascade of the two

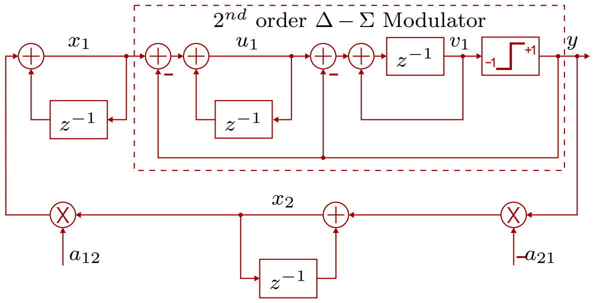

Fig. 1. Block diagram of the considered modulator–resonator. The parameters a_{12} and a_{21} determine the frequency of the generated sinusoid.

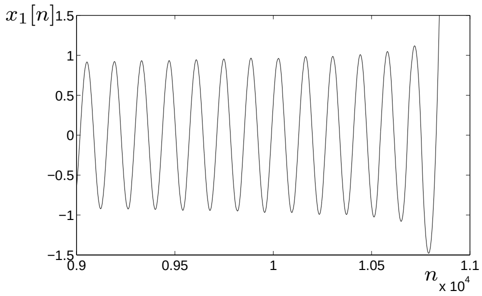

Fig. 2. Time waveform of $x_1[\cdot]$. The signal frequency is 22 kHz and the waveform diverges after about 20 real–time minutes.

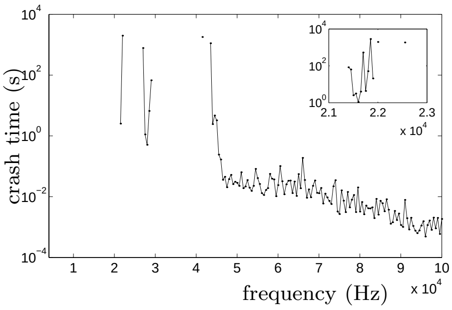

Fig. 3. Crash time of the oscillator as a function of the frequency. As $a_{12}a_{21}$ increases, the stability of the oscillator deteriorates.

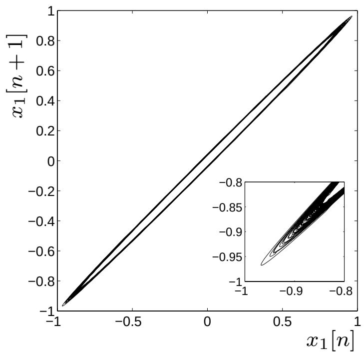

Fig. 4. Trajectory of the $x_1[\cdot]$ state variable on the phase plane. In the inset, a zoomed portion of such a trajectory is shown. Since the orbits are not superimposed, the corresponding waveform is not exactly periodic.

integrators outside the Δ–Σ modulator. The normalized frequency at which such new poles occur is $arccos(1 - a_{12}a_{21}/2)$, that is the frequency of the sinusoidal waveform that the system generates [4]. Unfortunately, the additive linear model is too coarse to cope with stability issues. In fact, it has been shown that, under certain initial conditions, the circuit becomes unstable and saturates quickly [5]. The unpredictable behavior of the oscillator is confirmed by the result of several simulations performed by the authors using floating point arithmetic over a period of about 3 hours of real–time continuous operation. This time interval, that is much longer than the oscillator would be required to work during a typical test process, has been evaluated by setting a clock frequency equal to 3.07 MHz and by counting a suitable number of clock cycles. As an example, in Fig. 2 the last $2 \cdot 10^3$ samples of the system state variable $x_1[\cdot]$ are plotted, by assuming a given set of initial conditions ($x_1[0] = 0.0$, $x_2[0] = 0.15265$, $u_1[0] = 0.0$, $v_1[0] = 0.0$, $y[0] = 1.0$). The values of a_{12} and a_{21} are chosen so that the frequency of the self–generated sinusoid is equal to 22 kHz. Note that, the waveform suddenly diverges after about 20 equivalent minutes of real–time simulation.

In order to appreciate the erratic behavior of the resonator–modulator under examination, in Fig. 3 it is shown the crash time of the oscillator as a function of the generated sinusoid frequency, evaluated considering a frequency resolution of 500 Hz. Missing points on the graph indicate that the system operated correctly over the given period of time. In the inset, it is shown the result of an additional simulation, carried out in the interval 21-23 kHz with a frequency resolution of 50 Hz. Note that, the finer resolution reveals other crash points, thus confirming the critical behavior of the system. It has been observed that a little variation in frequency can cause a sudden malfunction after an unpredictable time ranging from few milliseconds to several minutes of real–time operation. According to the additive linear model, the failure of this system can be considered as the consequence of the large amount of shaped noise power which is fed back into the resonator. The main consequence of the coarse quantization error is that the overall system is very sensitive to variations in the initial conditions of state variables. Sometimes, this behavior is referred to as chaotic [6]. This is further evidenced in Fig. 4, where the aperiodic patterns of the $x_1[\cdot]$ cyclic trajectory are shown on the phase plane. Unfortunately, an analytical stability assessment of the resonator–modulator appears to be excessively complicated to be solved analitically. For this reason, in the next section, the describing function method (DFM) is employed to perform a stability analysis based on the root locus.

III. A STABILITY ANALYSIS OF THE 1–BIT Δ–Σ RESONATOR–MODULATOR

The DFM is a classical technique to detect limit cycles in nonlinear systems [7]. This method is based on two main hypotheses:

1. the input signal of the nonlinear component is a sinusoid;
2. most of the power of the output signal of nonlinear element is associated with the fundamental harmonic.

According to the DFM, a nonlinear element in a system can be approximated by the ratio between the the output fundamental and the input sinusoid. Usually, this function depends on both the amplitude and the frequency of the sinusoidal input to the nonlinear element and it consists of a gain and a phase shift. In Δ–Σ modulators, the DFM allows complete prediction of the system stability only if the phase uncertainty introduced by the 1–bit quantizer is taken into account [8]. However, since for the circuit in Fig. 1 the phase of the self-generated sinusoid can be exactly computed [4], the phase uncertainty can be considered equal to 0. As a consequence, the 1–bit quantizer can be modeled in the z–domain simply by a global signal gain which, in the following, will be referred to as λ. This gain is time–varying and results from the ratio between the constant unit amplitude output and the variable amplitude input of the quantizer. According to this definition the gain varies from $\lambda \rightarrow +\infty$ when $v_1 \rightarrow 0$, to $\lambda \rightarrow 0$ when v_1 becomes very large. Thus, the system keeps on being nonlinear as the gain varies chaotically across samples. However, since the modulator–resonator can be supposed to be linear during a single sampling period, the system can be described by the following space–state model:

$$\begin{bmatrix} x_2[n+1] \\ x_1[n+1] \\ u_1[n+1] \\ v_1[n+1] \end{bmatrix} = \boldsymbol{F} \cdot \begin{bmatrix} x_2[n] \\ x_1[n] \\ u_1[n] \\ v_1[n] \end{bmatrix} \qquad (2)$$

where,

$$\boldsymbol{F} \triangleq \begin{bmatrix} 1 & 0 & 0 & -a_{21}\lambda \\ a_{12} & 1 & 0 & -a_{12}a_{21}\lambda \\ a_{12} & 1 & 1 & -(1+a_{12}a_{21})\lambda \\ a_{12} & 1 & 1 & 1-(2+a_{12}a_{21})\lambda \end{bmatrix}. \qquad (3)$$

The z–transform of the free evolution of the system state is given by:

$$\begin{bmatrix} X_2(z) \\ X_1(z) \\ U_1(z) \\ V_1(z) \end{bmatrix} = z(z\boldsymbol{I}-\boldsymbol{F})^{-1} \cdot \begin{bmatrix} x_2[0] \\ x_1[0] \\ u_1[0] \\ v_1[0] \end{bmatrix} = \frac{1}{P(z)} \begin{bmatrix} A(z) \\ B(z) \\ C(z) \\ D(z) \end{bmatrix} \qquad (4)$$

where $P(z) = z^4 + [(2+a_{12}a_{21})\lambda - 4]z^3 + (6-5\lambda)z^2 + +4(\lambda-1)z - \lambda + 1$ is the characteristic polynomial of the system and $A(z)$, $B(z)$, $C(z)$, $D(z)$ are polynomials depending on the initial conditions $x_1[0]$, $x_2[0]$, $u_1[0]$, $v_1[0]$. The general expressions for $A(z)$, $B(z)$, $C(z)$, $D(z)$ are very involved and do not add insight into the system, so that they are not reported in this

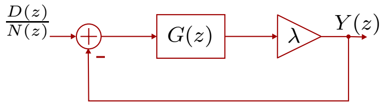

Fig. 5. Equivalent block diagram of the modulator–resonator. The 1–bit quantizer is modeled by the time–varying gain λ. The system initial conditions influence only the coefficients of $D(z)$, that is the amplitudes of the equivalent system input modes.

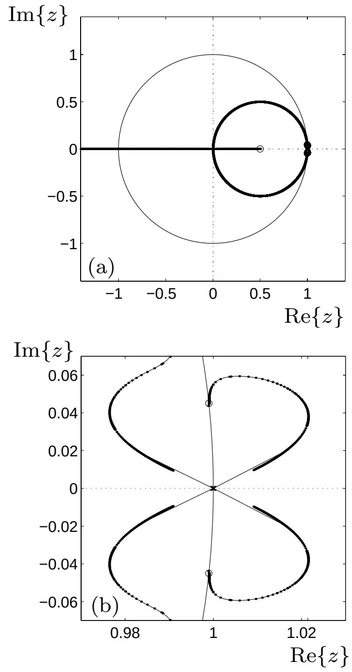

Fig. 6. Root locus of the resonator–modulator equivalent circuit when the generated sinusoid has a 22 kHz frequency. In (a) the full root locus is shown, while in (b) a zoomed portion of (a) is plotted around $z = (1,0)$.

paper. If the z–transform of the oscillator output is examined, it results that:

$$Y(z) \triangleq \lambda V_1(z) = \frac{\lambda D(z)}{P(z)} = \frac{D(z)}{N(z)} \cdot \frac{\lambda G(z)}{1+\lambda G(z)} \qquad (5)$$

where:

$$D(z) \triangleq v_1[0]z^4 + (a_{12}x_2[0] - 3v_1[0])z^3 + 3v_1[0]z^2 - v_1[0]z, \qquad (6)$$

$$N(z) \triangleq (2+a_{12}a_{21})z^3 - 5z^2 + 4z - 1, \qquad (7)$$

$$G(z) \triangleq \frac{N(z)}{(z-1)^4}. \qquad (8)$$

It can easily be shown that (5) is the algebraic expression corresponding to the parametric digital system reported in Fig. 5. Note that the particular expression of $D(z)$ reported in (6) has been obtained from (4) by assuming $u_1[0] = 0.0$ and $v_1[0] = 0.0$, as stated in section II. Of course, the system stability is related to the position of the system poles on the complex plane. Such positions can be graphically represented as a function of λ through the root locus technique [8]. If the amplitudes of the quantizer input signals are chosen so that all

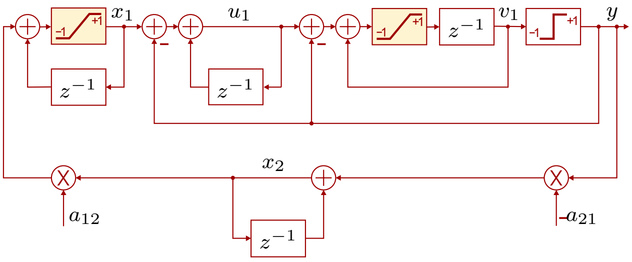

Fig. 7. Block diagram of the modified Δ–Σ modulator–resonator. Circuit stabilization is achieved through the 2 soft limiters highlighted in grey.

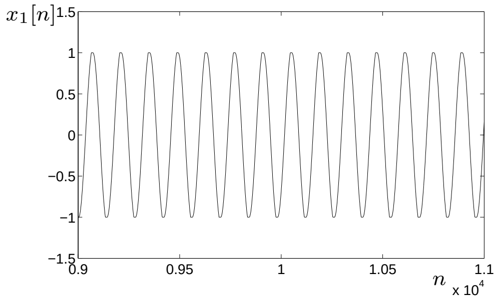

Fig. 8. Time waveform of $x_1[\cdot]$ at a frequency of 22 kHz. Due to the insertion of soft limiters, the waveform does not diverge after 20 minutes.

poles lie inside the unit circle, the whole circuit will be asymptotically stable [9]. Intersection points between the root locus and the unit circle imply the existence of limit cycles. The fundamental radian frequency of the self–generated waveform is given by the angle of these intersection points in polar coordinates. As an example, in Fig. 6(a) it is shown the root locus of the resonating system when the oscillation frequency is set at 22 kHz. In Fig. 6(b) it is plotted an enlarged view of the same root locus near the point $z = (1, 0)$. In order to validate the model employed to describe the system, several values of the quantizer gain deriving from a simulation of the circuit in Fig. 1 are superimposed (dotted lines) to the analytic root locus (continuous lines) of the equivalent circuit shown in Fig. 5. The unstable behavior of the oscillator is clearly highlighted by the pattern shown in Fig. 6(b). In fact, the time–varying poles associated with the oscillating modes tend to move outside the unit circle when λ decreases (i.e. when the amplitude of the within loop signal increases). Unfortunately, the chaotic sequence of gain values in the equivalent circuit makes it difficult to predict exactly when the failure will occur. Nevertheless, since the root locus of the system shows the same pattern regardless of self–generated frequency of the sinusoid, the limit cycle is always critically stable. This is confirmed by the fact that, increasing $x_2[0]$ beyond a certain threshold, the digital circuit saturates even at 5 kHz or less. To avoid the failure of the system, a simple solution is illustrated in Fig. 7. By inserting 2 soft limiters inside 2 Lossless Discrete Integrators, it is possible to bound λ, thus preventing the saturation of the oscillator. In Fig. 8 the last $2 \cdot 10^3$ samples of $x_1[\cdot]$ are plotted, by assuming the same initial conditions as those related to the waveform shown in Fig. 2. Values a_{12}, a_{21} are chosen so that the self–generated sinusoid frequency is equal to 22 kHz. Other simulations, performed for several frequency values, did not evidence any system malfunction. However, this increase in stability is traded for a reduction in the quality of the generated harmonic signal.

IV. Conclusions

The correct operation of Δ–Σ resonator–modulators depends on the stability of the self–generated limit cycles. In this paper, by modeling the 1–bit quantizer as a gain, it has been applied the root locus technique to assess the stability of an oscillator based on a second–order Δ–Σ topology. From such an analysis it emerges that even if, under some conditions, an oscillation can sustain indefinitely, the modulator–resonator shown in Fig. 1 is intrinsically unstable because of the influence of the 1–bit quantizer gain on the poles of the system. Nevertheless, by inserting properly two soft limiters inside the loop, the sinusoidal limit cycle can be stabilized.

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