IMEKO 23rd TC3, 13th TC5 and 4th TC22 International Conference
30 May to 1 June, 2017, Helsinki, Finland

VALIDITY OF EXTRAPOLATION BASED ON POLYNOMIAL APPROXIMATIONS

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Abstract: This paper describes a method to find an appropriate extrapolation with its corresponding uncertainty for any sensor in its full working range with the information provided by calibrations in partial ranges lower than the full working range. It is based on the assumption that the relation between the sensor input and output can be described by means of polynomial approximations. The method includes several conditions that are established in order to ensure a valid approximation for the extrapolation. The work is a contribution to the EMRP project "Force traceability within the meganewton range".

Keywords: Extrapolation, sensor, polynomial approximation, partial range.

1. INTRODUCTION

This paper describes the work that has been performed as a contribution to the Joint Research Project SIB63, "Force traceability within the meganewton range", under the EMRP [1].

This project addresses the needs of the European Industry in the field of force metrology. In Europe there are many industrial applications for force measurement and there is an increased demand for higher nominal forces.

The development of high range force standard machines is technically difficult and extremely expensive, and if possible, the corresponding force transducers calibration is expensive and time consuming. Consequently, as a part of this project, reliable methods for the extrapolation of measurement results for the range exceeding the traceable measuring range of calibration laboratories have to be developed to cover forces up to 50 MN. The methods to be developed in this JRP for extrapolating calibration results to values higher than 15 MN force, including evaluation of the associated uncertainties, will have an important impact.

Although the origin of this work has been the project previously mentioned, the method proposed in this paper is universal. It can be used for any sensor and any quantity as it is only based in mathematics and statistics. The only condition is the sensor working principle ensures its behaviour for the full range can be determined by means of a polynomial approximation.

The method is based on the idea of performing calibrations of the sensor under study in several partial ranges lower than the sensor range and extrapolating the observed behaviour to the full range. In principle, this is a straightforward way to avoid the sensor calibration in its full range.

2. METHOD DESCRIPTION

2.1 Approximations

The first issue to be considered for the proposed method is the way to perform an approximation for the relation between input and output of the sensor. This approximation will be performed by algorithms based on the "minimum χ^2" approach, that is, the parameters of the model function for the approximation $y = f(x)$ are determined so that expression (1) is fulfilled with n the number of test points and u_j the corresponding measurement uncertainties.

$$\chi^2_{\text{obs}} = \sum_{j=1}^{n} \frac{\left[f(x_j) - y_j\right]^2}{u_j^2} = \text{minimum} \tag{1}$$

From the observed chi-squared value χ^2_{obs}, if condition (2) is met with the degrees of freedom $\upsilon = n - n_{\text{par}}$ and being n_{par} the number of parameters of $y = f(x)$ to be determined, it is justified to assume the form of the model function $y = f(x)$ to be mathematically consistent with the data underlying the approximation.

$$\chi^2_{\text{obs}} \le \upsilon \tag{2}$$

This is a simplified case of [3] and the model function to be considered will be a polynomial as expressed in (3).

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n_a} x^{n_a} \tag{3}$$

The degree n_{a} of the polynomial should be chosen such that $n_{\text{par}} = n_{\text{a}} + 1 \le n/2$.

This calculation is best performed by matrix calculation. Let

- $\boldsymbol{X}_{(n \times n_{\text{par}})}$ be a matrix whose n rows are $(1, x_j, x_j^2, \ldots, x_j^{n_a})$,

- $\boldsymbol{a}_{(n_{\text{par}} \times 1)}$ be a column vector whose components are the coefficients a_0, a_1,..., a_{n_a} to be determined of the approximation polynomial,
- $y_{(n \times 1)}$ be a column vector whose components are the y_j,
- $\boldsymbol{V}(\boldsymbol{y})_{(n \times n)}$ be the variance-covariance matrix of $\boldsymbol{y}$. $\boldsymbol{V}(\boldsymbol{y})$ is given by $\boldsymbol{V}(\boldsymbol{y}) = \boldsymbol{V}_{\text{ref}} + \boldsymbol{V}_{\text{cal}} + \boldsymbol{V}_{\text{approx}}$ where
  - $\boldsymbol{V}_{\text{ref}}$ is the variance-covariance matrix associated with the reference values. Being $\boldsymbol{s}_{\text{ref}}$ the column vector of the uncertainties $u_{j,\text{ref}}$ and considering reasonably high correlation among the reference values, this variance-covariance matrix will be given by $\boldsymbol{V}_{\text{ref}} = \boldsymbol{s}_{\text{ref}} \boldsymbol{s}_{\text{ref}}^{\mathsf{T}}$;
  - $\boldsymbol{V}_{\text{cal}}$ is a diagonal matrix whose diagonal elements are $u_{j,\text{cal}}^2$, which are given by the uncertainty components that only depend on the sensor;
  - $\boldsymbol{V}_{\text{approx}}$ is an additional diagonal matrix given by $\boldsymbol{V}_{\text{approx}} = u_{\text{approx}}^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix and u_{approx} is an uncertainty due to the model. This contribution is considered in order to take into account the model inadequacy.

The weighting matrix $\boldsymbol{P}$ is given by expression (4),

$$\boldsymbol{P} = \boldsymbol{V}(\boldsymbol{y})^{-1} \tag{4}$$

so the coefficients $a_0, a_1, \ldots$ are found by solving the normal equations in (5).

$$\boldsymbol{X}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{X} \boldsymbol{a} - \boldsymbol{X}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{y} = \boldsymbol{0} \tag{5}$$

The solution will then be given by equation (6).

$$\hat{\boldsymbol{a}} = \left(\boldsymbol{X}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{y} \tag{6}$$

The n residuals $v_j = f(x_j) - y_j$ are comprised in vector (7),

$$\boldsymbol{v} = \boldsymbol{X} \hat{\boldsymbol{a}} - \boldsymbol{y} \tag{7}$$

and χ_{obs}^2 is obtained by expression (8).

$$\chi_{\text{obs}}^2 = \boldsymbol{v}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{v} \tag{8}$$

Provided condition (2) is met for χ_{obs}^2 according to (8), the variances and covariances for the coefficients a_i will be given by matrix (9).

$$\boldsymbol{V}(\hat{\boldsymbol{a}}) = \left(\boldsymbol{X}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{X}\right)^{-1} \tag{9}$$

Generally when condition (2) is not met, one of these procedures may be applied:
a) Repeat the approximation with an approximating polynomial of higher degree n_a, as long as $n_a + 1 \le n/2$, or
b) Repeat the approximation after increasing $\boldsymbol{V}_{\text{approx}}$.
The estimated values $\hat{y}_j$ for the n points x_j .derived from the approximation function are comprised in vector (10),

$$\hat{\boldsymbol{y}} = \boldsymbol{X} \hat{\boldsymbol{a}} \tag{10}$$

and the corresponding uncertainties can be determined from expression (11).

$$u^2(\hat{y}_j) = \left\{\text{diag}\left[\boldsymbol{X} \boldsymbol{V}(\hat{\boldsymbol{a}}) \boldsymbol{X}^{\mathsf{T}}\right]\right\}_j \tag{11}$$

2.2 Validity for extrapolation

This approximation will be performed for each partial range i under study, $y = f_i(x)$. This is obvious that the function $f(x)$ should not depend on the range, as it represents the behaviour of the sensor.

In order to be able to perform a valid extrapolation, a subset of s partial ranges has to be found for which the corresponding approximations fulfil the conditions established below.

1) For all these partial ranges the corresponding approximations will be polynomials with the same degree n_a, this is for each partial range i under study the approximation function will follow expression (12).

$$f_i(x) = \hat{a}_{0i} + \hat{a}_{1i} x + \hat{a}_{2i} x^2 + \ldots + \hat{a}_{n_a i} x^{n_a} \tag{12}$$

As a consequence, condition (2) should be met for all these approximating polynomials obtained for the subset of s partial ranges.

The other conditions will be evaluated in the range under study for the extrapolation (normally the sensor full range), which will be characterised for a set of values $\{x_k\}$, $k = 1, \ldots, m$.

2) One of the partial ranges will be chosen as the reference range, so that, if all these conditions are met, the corresponding approximation function for this range $y = f_r(x)$ (reference approximation) will be a valid extrapolation function. The comparisons of the approximation function for each partial range i, $i = 1, \ldots, s$ of the subset with the reference approximation will have to be mathematically consistent. In order to ensure this condition, the following procedure has to be performed. Firstly, the residuals $v_{i \leftrightarrow r, j} = f_i(x_k) - f_r(x_k)$ are determined; secondly $\chi_{i \leftrightarrow r}^2$ is obtained by expression (13).

$$\chi_{i \leftrightarrow r}^2 = \boldsymbol{v}_{i \leftrightarrow r}^{\mathsf{T}} \boldsymbol{P}_{i \leftrightarrow r} \boldsymbol{v}_{i \leftrightarrow r} \tag{13}$$

The matrix $\boldsymbol{P}_{i\leftrightarrow r}$ is defined as $\boldsymbol{P}_{i\leftrightarrow r} = [\boldsymbol{V}_i(\boldsymbol{y}) + \boldsymbol{V}_r(\boldsymbol{y})]^{-1}$, where $\boldsymbol{V}_i(\boldsymbol{y})$, $i = 1,...,s$ and r, is the variance-covariance matrix that takes into consideration the relative uncertainties obtained during the calibration in the partial range i, but applied to the full range under study $\{y_k\}$, $k = 1,...,m$ (further clarification will be provided in section 3). Finally it is checked that condition (14) is fulfilled for each partial range i of the subset.

$$\chi^2_{i\leftrightarrow r} \le m - n_{\text{par}} \tag{14}$$

3) A further condition will be that the variable (15) follows a chi-squared distribution with *s* degrees of freedom.

$$\chi^2_{\text{approx}} = \sum_{i=1}^{s} \chi^2_{i\leftrightarrow r} \tag{15}$$

As a consequence, it is a requirement that condition (16) is also met.

$$\chi^2_{\text{approx}} \le s \tag{16}$$

4) The last condition has to do with the fact of obtaining a coherent result in the range under study for extrapolation. As expressed in (17), for each x_k in the range under study for extrapolation, the uncertainty for each y_k obtained with the reference approximation has to be equal or higher than the corresponding uncertainty obtained from the calibration in the reference range.

$$\{\text{diag}[\boldsymbol{X}\boldsymbol{V}(\hat{\boldsymbol{a}})\boldsymbol{X}^{\mathsf{T}}]\}_k \ge \{\text{diag}[\boldsymbol{V}_r(\boldsymbol{y})]\}_k \tag{17}$$

Under the conditions that have been established previously, it is justified to assume the approximation function $y = f_r(x)$ as a valid extrapolation function.

In this method, all the approximating polynomials, as long as $n_{\text{a}} + 1 \le n/2$, are studied at the same time. In order to find the valid subset of partial ranges with the appropriate approximation polynomials, initially u_{approx} is set to zero. If any of the conditions is not fulfilled, u_{approx} is enlarged in an iterative way until all the conditions are fulfilled for a certain n_{a}.

2.3 Uncertainty in use

The previous procedure provides a valid approximation for the extrapolation with its corresponding uncertainty. When this solution has to be used during the further use of the sensor, it is clear that equation (18) will be used, where $\tilde{\boldsymbol{x}}$ is a column vector whose elements are $(1, \tilde{x}, \tilde{x}^2, \tilde{x}^3, ..., \tilde{x}^{n_{\text{a}}})^{\mathsf{T}}$ and $\hat{\boldsymbol{a}}_r$ is the column vector with the estimated coefficients for the approximation polynomial in the reference range r.

$$\tilde{y} = f_r(\tilde{x}) = \tilde{\boldsymbol{x}}^{\mathsf{T}}\hat{\boldsymbol{a}}_r \tag{18}$$

Then the uncertainty in use will be obtained from expression (19), where $\tilde{\boldsymbol{x}}'$ is a column vector whose elements are the derivatives $(0, 1, 2\tilde{x}, 3\tilde{x}^2, ..., n_{\text{a}}\tilde{x}^{n_{\text{a}}-1})^{\mathsf{T}}$.

$$u^2(\tilde{y}) = (\tilde{\boldsymbol{x}}'^{\mathsf{T}}\hat{\boldsymbol{a}}_r)\boldsymbol{V}(\tilde{x})(\tilde{\boldsymbol{x}}'^{\mathsf{T}}\hat{\boldsymbol{a}}_r)^{\mathsf{T}} + \tilde{\boldsymbol{x}}^{\mathsf{T}}\boldsymbol{V}(\hat{\boldsymbol{a}}_r)\tilde{\boldsymbol{x}} \tag{19}$$

As $u(\tilde{x})$ takes into account only the standard uncertainties associated the sensor itself, $\boldsymbol{V}(\tilde{x}) = u^2(\tilde{x})\boldsymbol{I}$ and the first term on the right-hand side simplifies, as all 3 matrices are only one dimensional, to expression (20).

$$\begin{aligned}&(\tilde{\boldsymbol{x}}'^{\mathsf{T}}\hat{\boldsymbol{a}})\boldsymbol{V}(\tilde{x})(\tilde{\boldsymbol{x}}'^{\mathsf{T}}\hat{\boldsymbol{a}})^{\mathsf{T}} = \\ &= (\hat{a}_1 + 2\hat{a}_2\tilde{x} + 3\hat{a}_3\tilde{x}^2 + ... + n_{\text{a}}\hat{a}_{n_{\text{a}}}\tilde{x}^{n_{\text{a}}-1})^2 u^2(\tilde{x})\end{aligned} \tag{20}$$

3. EXAMPLE

During the project this procedure was tested with real examples of force transducers connected to their corresponding amplifiers. One case is included here as example. This transducer has been calibrated according to option C of the international standard ISO 376 [4] in several partial ranges, but also for the full capacity, so that a straightforward validation has been possible. The mathematical consistency of the obtained extrapolation functions with the data for the full range has been checked at 95.45 % significance level.

In this example the extrapolation of a 2 MN transducer has been studied. It has been calibrated in a hydraulic force machine with 10^{-4} as relative standard uncertainty. The calibrations have been performed in four ranges: 25 %, 30 %, 50 % and 75 % capacity, as well as for full capacity for validation purposes. In tables 1 and 2 the results for the different calibrations are included.

The uncertainty contributions that have been considered include the uncertainty due to the force standard machine, which provides the reference values, and the uncertainties components inherent to the transducer behaviour: resolution, effect of temperature variation in the transducer sensitivity (the environmental conditions during all the calibration tests were (20 ± 1) ºC), drift of the zero output, repeatability (variability in the results without changing the position of the transducer), reproducibility (variability in the results when the position of the transducer is rotated along its axis) and reversibility (variability when comparing results between increasing and decreasing forces). More details about the uncertainty determination of the transducer are included in [2] and [4]. For the uncertainty in use of the transducers all the uncertainty components inherent to the transducer, which have been mentioned previously, have been taken into consideration.

The transducer has been calibrated in 10 set points for each partial calibration range, so $n = 10$ and, consequently, the maximum valid degree for the approximation polynomial will be $n_{\mathrm{a}} = 5$. On the other hand, the calibration in the full range has also been performed in 10 set points, so $m = 10$.

Table 1. Results for the partial ranges: 25 %, 30 % and 50 % capacity.

25 %			30 %			50 %		
F /kN	V /mV/V	w /%	F /kN	V /mV/V	w /%	F /kN	V /mV/V	w /%
50	0.047 552	0.026 5	60	0.057 064	0.027 1	100	0.104 599	0.024 6
100	0.095 114	0.022 5	120	0.114 132	0.024 6	200	0.209 178	0.023 8
150	0.142 677	0.022 3	180	0.171 199	0.022 4	300	0.313 698	0.023 1
200	0.190 228	0.021 8	240	0.228 242	0.021 4	400	0.418 170	0.022 9
250	0.237 769	0.021 8	300	0.285 262	0.020 9	500	0.522 607	0.022 9
300	0.285 285	0.021 6	360	0.342 279	0.020 8	600	0.626 994	0.022 9
350	0.332 800	0.021 3	420	0.399 268	0.020 8	700	0.731 337	0.022 9
400	0.380 304	0.021 2	480	0.456 251	0.020 7	800	0.835 644	0.022 9
450	0.427 798	0.021 2	540	0.513 217	0.020 6	900	0.939 942	0.022 9
500	0.475 284	0.020 9	600	0.570 166	0.020 6	1 000	1.044221	0.022 9

Table 2. Results for the partial range 75 % capacity and full capacity.

75 %			100 %		
F /kN	V /mV/V	w /%	F /kN	V /mV/V	w /%
150	0.142 612	0.033 1	200	0.190 081	0.030 3
300	0.285 163	0.025 4	400	0.380 039	0.024 7
450	0.427 626	0.023 4	600	0.569 862	0.022 9
600	0.570 004	0.022 9	800	0.759 557	0.022 2
750	0.712 309	0.022 7	1 000	0.949 159	0.022 1
900	0.854 544	0.022 7	1 200	1.138 702	0.022 1
1 050	0.996 749	0.022 7	1 400	1.328 131	0.022 2
1 200	1.138 919	0.022 7	1 600	1.517 448	0.022 1
1 350	1.281 009	0.022 7	1 800	1.706 639	0.022 1
1 500	1.423 045	0.022 7	2 000	1.895 674	0.022 2

In the example the different applied forces will be indicated as F and the corresponding outputs of the transducers will be indicated as V. The corresponding combined relative standard uncertainties will be indicated as w. These relative standard uncertainties will be used to obtain the diagonal terms for $\boldsymbol{V}_{\mathrm{ref}} + \boldsymbol{V}_{\mathrm{cal}}$; the non diagonal terms will be obtained taking into consideration the relative standard uncertainty of the corresponding force machine. The contribution to uncertainty due the model inadequacy will be taken as a relative term also, w_{approx}. As a consequence, for the study of each approximation function, in order to get each term of the variance-covariance matrix $\boldsymbol{V}(\boldsymbol{y})$, $[\boldsymbol{V}(\boldsymbol{y})]_{a,b}$, these contributions will have to be multiplied by the corresponding term $x_{j_a} x_{j_b}$, $j = 1,...,n$. The previous considerations will also be used to obtain each term of the variance-covariance matrices $\boldsymbol{V}_i(\boldsymbol{y})$, $i = 1,..., s$ and r, but, in these cases, the contributions will have to be multiplied by the corresponding terms $x_{k_a} x_{k_b}$, $k = 1,..., m$.

Two cases will be studied: the extrapolation that relates the sensor deflection V as a function of the force F, $V = f_{F \to V}(F)$ and vice versa $F = f_{V \to F}(V)$. For all the approximation functions it has always been considered $a_0 = 0$ because the deflection caused by a force is defined as the difference between the output under this force and the output without force according to ISO 376 [4].

In principle the reference range considered has been the 75 % capacity. When applying this proposed method it was found that the approximation function for the 50 % capacity partial range was not mathematically consistent with the approximation functions for the other partial ranges. The solutions are included in table 3 and 4.

An additional interest of this example is it also includes the calibration for the sensor full capacity, which allows a validation of the method. In order to perform this validation, condition (2) is checked taking into account the calibration results for the full range and the reference approximation considered.

Table 3. Summary description of the applied method including its partial solutions for the relation force F versus deflection V and vice versa.

$V = f_{F \to V}(F)$	$F = f_{V \to F}(V)$
$w_{\mathrm{approx}} = 1.23 \times 10^{-3}$	$w_{\mathrm{approx}} = 7.1 \times 10^{-4}$
Approximations	
Units for a_1 : (mV/V)/kN Units for a_2 : (mV/V)/kN^2	Units for a_1 : kN /(mV/V) Units for a_2 : kN /(mV/V)^2
$f_{75\%}(F) = 9.510 \times 10^{-4} F - 1.548 \times 10^{-9} F^2$	$f_{75\%}(V) = 1051.571 V + 1.809 V^2$
$f_{50\%}(F) = 1.046 \times 10^{-3} F - 2.075 \times 10^{-9} F^2$	$f_{50\%}(V) = 955.792 V + 1.820 V^2$
$f_{30\%}(F) = 9.513 \times 10^{-4} F - 1.626 \times 10^{-9} F^2$	$f_{30\%}(V) = 1051.175 V + 1.899 V^2$
$f_{25\%}(F) = 9.513 \times 10^{-4} F - 1.235 \times 10^{-9} F^2$	$f_{25\%}(V) = 1051.146 V + 1.813 V^2$
1) Condition (2), $n - n_{\mathrm{par}} = 8$	
$\chi^2_{\mathrm{obs},75\%} = 0.013\,2$ (fulfilled)	$\chi^2_{\mathrm{obs},75\%} = 0.036\,0$ (fulfilled)
$\chi^2_{\mathrm{obs},50\%} = 0.006\,5$ (fulfilled)	$\chi^2_{\mathrm{obs},50\%} = 0.018\,2$ (fulfilled)
$\chi^2_{\mathrm{obs},30\%} = 0.034\,7$ (fulfilled)	$\chi^2_{\mathrm{obs},30\%} = 0.097\,4$ (fulfilled)
$\chi^2_{\mathrm{obs},25\%} = 0.053\,7$ (fulfilled)	$\chi^2_{\mathrm{obs},25\%} = 0.197\,6$ (fulfilled)
2) Condition (14), $m - n_{\mathrm{par}} = 8$	
$\chi^2_{50\% \leftrightarrow 75\%} = 30\,240.16$ (not fulfilled)	$\chi^2_{50\% \leftrightarrow 75\%} = 63\,546.09$ (not fulfilled)
$\chi^2_{30\% \leftrightarrow 75\%} = 0.25$ (fulfilled)	$\chi^2_{30\% \leftrightarrow 75\%} = 0.65$ (fulfilled)
$\chi^2_{25\% \leftrightarrow 75\%} = 1.73$ (fulfilled)	$\chi^2_{25\% \leftrightarrow 75\%} = 1.34$ (fulfilled)
3) Condition (16), $s = 2$	
$\chi^2_{\mathrm{approx}} = 1.98$ (fulfilled)	$\chi^2_{\mathrm{approx}} = 1.99$ (fulfilled)
4) Condition (17) is fulfilled from the comparison of table 2 and table 4	
Validation, $m - n_{\mathrm{par}} = 8$	
$\chi^2_{100\% \leftrightarrow 75\%} = 0.127$ (fulfilled)	$\chi^2_{100\% \leftrightarrow 75\%} = 0.328$ (fulfilled)

Table 4. Coefficients for the extrapolation and its corresponding variance-covariance matrix for the relation force F versus deflection V and vice versa (reference range = 75 % capacity). The relative standard uncertainties in the full range directly derived from the extrapolation (11) and in use (19) are also included.

$V = f_{F \to V}(F)$			$F = f_{V \to F}(V)$		
$\hat{\boldsymbol{a}}^{\mathsf{T}} = (9.510 \times 10^{-4} \quad -1.548 \times 10^{-9})$			$\hat{\boldsymbol{a}}^{\mathsf{T}} = (1\,051.571 \quad 1.809)$		
$\boldsymbol{V}(\hat{\boldsymbol{a}}) = \begin{pmatrix} 6{,}686 \times 10^{-13} & -6{,}274 \times 10^{-16} \\ -6{,}273 \times 10^{-16} & 7{,}571 \times 10^{-19} \end{pmatrix}$			$\boldsymbol{V}(\hat{\boldsymbol{a}}) = \begin{pmatrix} 0.305 & -0.295 \\ -0.295 & 0.373 \end{pmatrix}$		
F /kN	$u_{\mathrm{rel}}(\mathrm{approx})$	$u_{\mathrm{rel}}(\mathrm{use})$	V /mV/V	$u_{\mathrm{rel}}(\mathrm{approx})$	$u_{\mathrm{rel}}(\mathrm{use})$
200	0.070 %	0.077 %	0.190 081	0.043 %	0.053 %
400	0.056 %	0.061 %	0.380 039	0.035 %	0.042 %
600	0.046 %	0.050 %	0.569 862	0.029 %	0.036 %
800	0.041 %	0.046 %	0.759 557	0.026 %	0.033 %
1 000	0.044 %	0.048 %	0.949 159	0.027 %	0.034 %
1 200	0.053 %	0.057 %	1.138 702	0.033 %	0.038 %
1 400	0.066 %	0.069 %	1.328 131	0.040 %	0.045 %
1 600	0.082%	0.084 %	1.517 448	0.049 %	0.053 %
1 800	0.098%	0.100 %	1.706 639	0.059 %	0.062 %
2 000	0.115%	0.117 %	1.895 674	0.069 %	0.072 %

In order to provide a way to demonstrate the robustness of the method, the procedure has been repeated four more times with the only difference of choosing the other partial ranges as reference range. Tables 5 to 7 include a summary of the results with the same units as before. The validation condition (2) has also been checked for each reference range considered in order to ensure the mathematical consistency of the reference approximation function with the calibration results for the full capacity. This condition has been fulfilled in all cases.

It has been found that the approximation function for 50 % capacity is never mathematically consistent with the others, it does not matter what partial range is chosen as the reference range, except when the 50 % capacity itself is taken as the reference range. In this case the relative uncertainty due to the model inadequacy, w_{approx}, which is required for consistency, is huge. Consequently, the relative standard uncertainties in the full range directly derived from the extrapolation, $u_{\text{rel}}(\text{approx})$, and the relative standard uncertainties in use in the full range, $u_{\text{rel}}(\text{use})$, are abnormally high. The obtaining of these abnormally high uncertainties obviously means there is a problem of consistency with these data and it would be advisable to use a different partial range as reference range instead.

Table 5. Coefficients for the extrapolation and its corresponding variance-covariance matrix for the relation force F versus deflection V and vice versa (reference range = 50 % capacity). The relative standard uncertainties in the full range directly derived from the extrapolation, and in use are also included. The relative standard uncertainties directly derived from the extrapolation (11), in use (19) and due to the model inadequacy are also included.

$V = f_{F\to V}(F)$			$F = f_{V\to F}(V)$		
$\hat{a}^{\mathsf{T}} = (1.046\times10^{-3} \quad -2.075\times10^{-8})$			$\hat{a}^{\mathsf{T}} = (955.792 \quad 1.820)$		
$V(\hat{a}) = \begin{pmatrix} 2.560\times10^{-8} & -3.656\times10^{-11} \\ -3.656\times10^{-11} & 6.643\times10^{-14} \end{pmatrix}$			$V(\hat{a}) = \begin{pmatrix} 18\,136.249 & -24\,814.79 \\ -24\,814.79 & 43\,208.98 \end{pmatrix}$		
$w_{\text{approx}} = 2.24\times10^{-1}$			$w_{\text{approx}} = 2.06\times10^{-1}$		
F /kN	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$	V /mV/V	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$
200	12.3 %	12.3 %	0.190 081	9.6 %	9.6%
400	8.8 %	8.8 %	0.380 039	7.1 %	7.1 %
600	7.9 %	7.9 %	0.569 862	5.9 %	5.9 %
800	10.3 %	10.3 %	0.759 557	7.0 %	7.0 %
1000	14.5 %	14.5 %	0.949 159	9.5 %	9.5 %
1 200	19.3 %	19.3 %	1.138 702	12.6 %	12.6 %
1 400	24.4 %	24.4 %	1.328 131	16.0 %	16.0 %
1 600	29.6 %	29.6 %	1.517 448	19.5 %	19.5 %
1 800	34.9 %	34.9 %	1.706 639	23.1 %	23.1 %
2 000	40.2 %	40.2 %	1.895 674	26.7 %	26.7 %

On the other hand, from the comparison among the results obtained when taking 75 % capacity, 30 % capacity or 25 % capacity as reference ranges, it can be concluded that the use of a shorter reference range takes to a higher relative uncertainty due to the model inadequacy, w_{approx}. This fact makes the relative standard uncertainties in the full range directly derived from the extrapolation, $u_{\text{rel}}(\text{approx})$, and, consequently, the relative standard uncertainties in use in the full range, $u_{\text{rel}}(\text{use})$, to be higher also. This is expected because the shorter the reference range is there is less information available about the sensor behaviour for its full capacity.

Table 6. Coefficients for the extrapolation and its corresponding variance covariance matrix for the relation force F versus deflection V and vice versa (reference range = 30 % capacity). The relative standard uncertainties directly derived from the extrapolation (11), in use (19) and due to the model inadequacy are also included.

$V = f_{F\to V}(F)$			$F = f_{V\to F}(V)$		
$\hat{a}^{\mathsf{T}} = (9.513\times10^{-4} \quad -1.631\times10^{-9})$			$\hat{a}^{\mathsf{T}} = (1051.175 \quad 1.899)$		
$V(\hat{a}) = \begin{pmatrix} 2.484\times10^{-13} & -5.684\times10^{-16} \\ -5.684\times10^{-16} & 1.710\times10^{-18} \end{pmatrix}$			$V(\hat{a}) = \begin{pmatrix} 0.304 & -0.732 \\ -0.732 & 2.318 \end{pmatrix}$		
$w_{\text{approx}} = 7.2\times10^{-4}$			$w_{\text{approx}} = 7.2\times10^{-4}$		
F /kN	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$	V /mV/V	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$
200	0.031 %	0.040 %	0.190 081	0.031 %	0.040 %
400	0.027 %	0.035 %	0.380 039	0.027 %	0.035 %
600	0.045 %	0.049 %	0.569 862	0.045 %	0.049 %
800	0.069 %	0.072 %	0.759 557	0.069 %	0.072 %
1 000	0.095 %	0.097 %	0.949 159	0.095 %	0.097 %
1 200	0.122 %	0.124 %	1.138 702	0.122 %	0.123 %
1 400	0.149 %	0.150 %	1.328 131	0.148 %	0.150 %
1 600	0.177 %	0.178 %	1.517 448	0.175 %	0.176 %
1 800	0.204 %	0.205 %	1.706 639	0.202 %	0.203 %
2 000	0.231 %	0.232 %	1.895 674	0.229 %	0.230 %

Table 7. Coefficients for the extrapolation and its corresponding variance covariance matrix for the relation force F versus deflection V and vice versa (reference range = 25 % capacity). The relative standard uncertainties directly derived from the extrapolation (11), in use (19) and due to the model inadequacy are also included.

$V = f_{F\to V}(F)$			$F = f_{V\to F}(V)$		
$\hat{a}^{\mathsf{T}} = (9.513\times10^{-4} \quad -1.291\times10^{-9})$			$\hat{a}^{\mathsf{T}} = (1051.202 \quad 1.500)$		
$V(\hat{a}) = \begin{pmatrix} 6.286\times10^{-13} & -1.769\times10^{-15} \\ -1.769\times10^{-16} & 6.418\times10^{-18} \end{pmatrix}$			$V(\hat{a}) = \begin{pmatrix} 1.150 & -3.423 \\ -3.423 & 13.077 \end{pmatrix}$		
$w_{\text{approx}} = 1.18\times10^{-3}$			$w_{\text{approx}} = 1.46\times10^{-3}$		
F /kN	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$	V /mV/V	$u_{\text{rel}}(\text{approx})$	$u_{\text{rel}}(\text{use})$
200	0.044 %	0.054 %	0.190 081	0.054 %	0.062 %
400	0.052 %	0.058 %	0.380 039	0.063 %	0.068 %
600	0.095 %	0.099 %	0.569 862	0.116 %	0.119 %
800	0.145 %	0.148 %	0.759 557	0.177 %	0.179 %
1 000	0.197 %	0.199 %	0.949 159	0.241 %	0.242 %
1 200	0.250 %	0.251 %	1.138 702	0.305 %	0.306 %
1 400	0.303 %	0.304 %	1.328 131	0.369 %	0.370 %
1 600	0.356 %	0.357 %	1.517 448	0.433 %	0.434 %
1 800	0.409 %	0.410 %	1.706639	0.498 %	0.498 %
2 000	0.463 %	0.463 %	1.895 674	0.562 %	0.563 %

4. SUMMARY AND CONCLUSIONS

This paper explains a methodology to find a valid extrapolation with its corresponding uncertainty for any sensor behaviour in its full working range based on calibration results obtained in shorter partial ranges. The method assumes the sensor working principle ensures its behaviour for its full range can be determined by means of a polynomial approximation. It is based on the fulfilment of several conditions to ensure the validity of the approximation function for the extrapolation in the sensor full range. The goal of this procedure is to avoid the calibration in the full range, which maybe very expensive or time-consuming, even impossible sometimes. The drawback is the uncertainty which is obtained as a result of the extrapolation will be increased if compared with the corresponding calibration uncertainty.

As an example, a real case for the extrapolation of force transducers calibration results has been included in this paper in order to illustrate the method performance and its validity.

During the application of this method it is recommended to study all the possible partial ranges as reference ranges, not only choose the highest partial range as the reference range. The reason is to avoid choosing the approximation function of a partial range that may not be consistent with the other approximation functions.

Additionally, there is an inverse relationship between the length of the reference range and the obtained uncertainties derived from the extrapolation: the longer the reference range is, the lower the expected uncertainties derived from the extrapolation will be. This is evident as more information is provided about the sensor behaviour in its whole range from a longer reference range.

The authors would like to thank the EMRP programme, which is jointly funded by the EMRP participating countries within EURAMET and the European Union.

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