Joint **IMEKO TC-1 & XXXIV MKM Conference 2002**
Wrocław, 8 - 12 September 2002

Jerzy M. KORCZYŃSKI*
Andrzej HETMAN

CALCULATION OF EXPANDED UNCERTAINTY

The main purpose of the paper is to present a method of calculation of the expanded uncertainty at required confidence level. It is elaborated as a piece of software with the main intention for metrology and quality engineering courses. The definition based algorithm is used and convolutions of different probability density distribution are precisely calculated. The uniform, triangular, normal and t-Student distributions are implemented in the software. One of the advantage is no limitation of encountered measurands
Keywords: expanded uncertainty, coverage factor, convolutions, e-teaching

1. THE CHALLENGE

The calculation of uncertainties is not an easy task not only for students but also for experienced professionals working in metrology. The Guide "Guide to expression of the uncertainty in measurements" is giving a general approach to evaluation of uncertainties. The new approach lasting since 1980 (Recommendation INC-1) developed later up to publication of Guide June 1992 of which main purpose was to promote full information on how uncertainty statements are arrived at and to provide a basis for the international comparison of measurement results.

The Guide concentrates on calculation of A-type uncertainties and the Welch-Satterthwaite formula is suggested for effective degrees of freedom calculation and then t-Student distribution for choosing the coverage factor. The B-uncertainties are merely mentioned and also the Welch-Satterthwaite formula is suggested for calculation of coverage factor for uniform and normal or t-Student distributions.

The results achieved with above mentioned methods in Guide are the estimations,

* Technical University of Łódź, Institute of Theoretical Electrotechnics, Metrology and Material Science, 90-924 Łódź, Poland, e-mail: jerzykor@ck-sg.p.lodz.pl

not accurate by comparison what is possible to get with a new numerical method elaborated by the authors. The calculations are based on convolution of distributions.

The only possible errors in calculation of expanded uncertainties arise due to assumptions for type of probability distributions not from approximation of *k*-factor or degrees of freedom.

2. THE ALGORITHM

The specified expanded uncertainty *U* also referred to as *overall uncertainty equal to half range of uncertainty interval with a certain level of confidence*, *p*, is given by.

$$U = k \cdot u_\mathrm{c} \tag{1}$$

where: *U* - is an expanded uncertainty, *k* - coverage factor chosen on the base of the desired level of confidence *p* to be associated with the interval *U*.

The combined standard uncertainty, u_c, is a geometrical sum of u_A - uncertainty calculated based on type A method and u_B - uncertainty calculated based on type B method:

$$u_\mathrm{c}^2 = u_\mathrm{A}^2 + u_\mathrm{B}^2 \tag{2}$$

u_A - standard uncertainty of type A, evaluated by statistical methods, estimation based on series of results,

u_B - standard uncertainty of type B, evaluated by other means than based on series of results.

While performing an indirect measurement, the uncertainties u_A and u_B may be expressed by several components with a different density distributions functions mainly of normal, t-Student distribution, uniform, and triangular shapes.

The users are calculating themselves the sensitivity coefficients $c_i = \frac{\partial f}{\partial x_\mathrm{i}}$ and $c_{i,j} = \frac{\partial f}{\partial x_\mathrm{i}} \frac{\partial f}{\partial x_\mathrm{j}}$, where $f(x_1, x_2, \ldots, x_N)$ is *Y* determined the measurand, which consists of N components of individually measured or defined quantities forming the overall measuring of $x_1, x_{2,} \ldots, x_N$ through a functional relation *f*, so:

$$Y = f\left(x_1, x_{2,} \ldots, x_N\right) \tag{4}$$

The complete formula describing combined uncertainty is as follows:

$$u_c^2(y) = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_i}\right]^2 \cdot u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u\left(x_i, x_j\right) \tag{5}$$

The evaluation of *k*-factor to calculate $U = ku_c$, where combined standard uncertainty, u_c, is calculated as Equ. (5) requires of choosing k - factor, for the combined distribution, or just deliberately chose of $k = 2$ for confidence level of $p = 0.95$ or 3 for $p = 0.99$.

The algorithm of direct numerical method of expanded uncertainty calculation is presented in Fig. 1 and the elaborated software is available at www.metrol.p.lodz.pl, the user can perform calculations via internet.

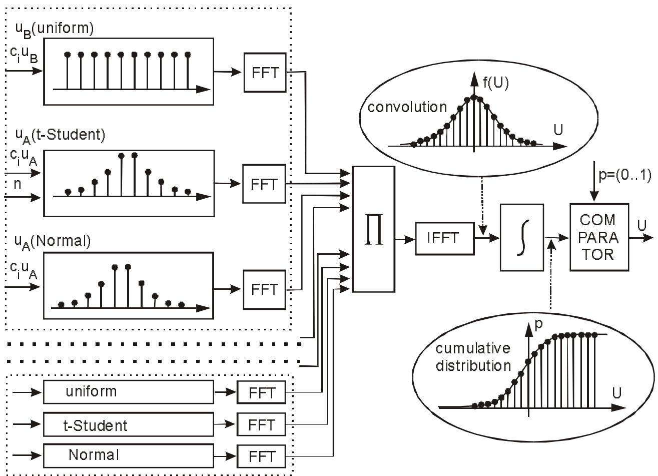


Fig.1 Block diagram of expanded uncertainty calculation. , FFT- Fast Fourier Transfor
Π- product, IFFT – Inverse Fast Fourier Transform

The piece of software for calculation of expanded uncertainty allows to use components of which distribution is uniform u_B, (triangular consists of two uniform distributions), t-Student distribution expressed by u_A and number of measurements n and normal probability density distributions, which is in software described u_A and number of measurements declared not infinitive but for example as $n = 1000000$.

For uniform distribution just for users confidence the standard deviation, u_B, is multiplied by square root of 3, so $\sqrt{3}u_B$, what is defining the bounders of error in theory of errors so popular in worst case calculations, before uncertainty was intr oduced.

The piece of software is based on calculation of convulsion of compound distributions, which describe each component and based on that expanded uncertainty, and finally, the expanded coefficient is just a result of quotient of expanded and combined uncertainties at desired confidence level.

The confidence level can be chosen by the software user.

3. EXAMPLE OF EXPANDED UNCERTAINTY CALULATION

A voltmeter of nominal value of 100 V (range 0-100 V) and basic accuracy (accuracy class) stated by manufacturer, to be of 0.5, was used. The 16 results were recorded (72.4; 73.2; 72.0, 73.6; 72.5; 72.9; 72.2; 72.2; 72.9; 73.4; 73.0; 72.6; 72.8; 73.6; 73.4; 72.8).

Evaluate the expanded uncertainty and present as $Y = y \pm U$ for two levels of confidence $p = 0.95$ and $p = 0.99$.

<u>Solution:-</u>

The combined uncertainty, u_c, consists of two components: u_A and u_B, which have to be evaluated by using: type A and B method. Type A method based on the statistical analysis of the series of results and the type B can be calculated using the information stated by instrument manufacturer. Combined uncertainty is a square root of the sum of the squares of u_A and u_B. The coverage factor, k, must be evaluated for convolution of type A and B distributions at the required level of confidence $p = 0.95$ and $p = 0.99$.

Three methods will be presented and compared there:

1. Deliberate choice of k = 2 for p = 0.95 and k = 3 for p = 0.99,
2. Calculation based on Welch-Satterthwaite formula.

Convolution of components (elaborated piece of software).

<u>First part: Uncertainty of Type A</u>.

The sample mean

$$\overline{V} = \frac{1}{n}\sum_{k=1}^{n} V_{\mathrm{k}} = \frac{1}{16}\sum_{k=1}^{16} V_{\mathrm{k}} = \frac{1165.5}{16} = 72.84375\,\mathrm{V} \approx 72.8438\,\mathrm{V}$$

The experimental standard deviation of $s_{\overline{V}}$ of the mean V, is the estimates of standard deviation $\sigma(\overline{V})$. Standard uncertainty is equal to the estimated standard deviation from the series of measurements. Therefore, $u_A(\overline{V}) = s(\overline{V})$.

Table 1. Calculation results quoted in the table

No	V_k	$V_k - \overline{V}$	$(V_k - \overline{V})^2$
	V	V	V^2
1.	72.4	-0,4437	0,1969
2.	73.2	0,3563	0,1269
3.	72.0	-0,8438	0,7120
4.	73.6	0,7562	0,5718
5.	72.5	-0,3438	0,1182
6.	72.9	0,0563	0,0032
7.	72.2	-0,6437	0,4143
8.	72.2	-0,6437	0,4143
9.	72.9	0,0563	0,0032
10.	73.4	0,5563	0,3095
11.	73.0	0,1563	0,0244
12.	72.6	-0,2438	0,0594
13.	72.8	-0,0438	0,0019
14.	73.6	0,7562	0,5718
15.	73.4	0,5563	0,3095
16.	72.8	-0,0438	0,0019
Σ	1165.5	0,0001	3,8392

Type A evaluation of the uncertainty:

$$u_A = s(\overline{V}) = \sqrt{\frac{1}{n(n-1)}\sum_{k=1}^{n}(V_k - \overline{V})^2} = \sqrt{\frac{1}{16(16-1)}3.8392} = 0.1265\,\text{V} \approx 0.126\,\text{V}$$

Three cases:

Three methods will be presented and compared there:

1. Deliberate choice of k=2 for p=0.95 and k=3 for p=0.99

   p = 0.95, k = 2, so U = 0.632 V ≈ 0.64 V

   p = 0.99, k = 3, so U = 0.948 V ≈ 0.95 V

2. Calculation based on Welch-Satterthwaite formula:

$$\nu_{eff} = \frac{u_c^4(V)}{\sum_{i=1}^{2}\frac{c_i^4 u_i^4(V)}{\nu_i}} = \frac{0.316^4}{\frac{0.126^4}{(16-1)} + \frac{0.289^4}{(\infty-1)}} = 593$$

p = 0.95, k = 1,97, so U = 0.622 V ≈ 0.63 V

p = 0.99, k = 2.59, so U = 0.819 V ≈ 0.82 V

3. Convolution of components (elaborated piece of software www.metrol.p.lodz.pl)

P = 0.95, k = 1,82, so U = 0.572 V ≈ 0.58 V

P = 0.99, k = 2.20, so U = 0.693 V ≈ 0.70 V

Comparison of results

Confidence level	$\frac{method\,"1"-method\,"3"}{method\,"3"}*100$	$\frac{method\,"2"-method\,"3"}{method\,"3"}*100$
-	%	%
0.95	11	8.7
0.99	37	18

4. CONCLUSIONS

The piece of software is dedicated to all calibration laboratories, company quality labs, quality engineering divisions, at the moment is used to teaching purposes. The usefulness of the software was tested at national calibration laboratory in Poland with a positive opinion.

The new virtual multimeter, in which an expanded uncertainty calculation in a real time was implemented had been elaborated as a engineering diploma project by Boguslaw Cielecki now graduated from Electrical and Electronic Engineering Department of the Technical University of Łódź, Łódź, Poland

REFERENCES

[1] GNIOTEK K., *Creative Measurement in View of the Scope of Metrology,* Measurement – Journal of the IMEKO, 1997, 20, 4, 259 – 266.

[2] KALUS-JĘCEK B., KUSMIEREK Z., *Wzorce wielkości elektrycznych ocena niepewności pomiaru,* Technical University of Łódź, 2000.

[3] KUBISA S., *O niektórych problemach dydaktyki podstaw metrologii,* VIII KKM, Warszawa 18-20.10.1995., Technical University of Warsaw, Prace naukowe, Konferencje, z. 4, t. I, s. 81 – 86.

[4] KUBISA S., *Podstawy metrologii,* Technical Unicversity of szczecin, Szczecin 1995.

[5] TURZENIECKA D., *Ocena niepewności wyniku pomiaru*, Technical Unicversity of Poznan, 1977

[6] TURZENIECKA D., *Analiza dokładności wybranych przybliżonych metod oceny niepewności,* Technical Unicversity of Poznan, 1978.

[7] EAL-R2 *Expression of the Uncertainty of Measurement in calibration*, Edition 1, April 1997.

[8] *Guide to the expression of uncertainty in measurement,* BIPM/ IEC/ IFCC/ ISO/ IUPAC/ IUPAP/ OIML, 1995.

Appendix 1 calculation for confidence level 0.99

The software requires: declaration of confidence level, all components of standard uncertainty of A and B type uncertainties, which are declared at www.metrol.p.lodz, after pressing enter, the users data are send to server, and after calculation values are returned back to users. Just to speed up transmission a screen of user can by a little different as shown below.

Table

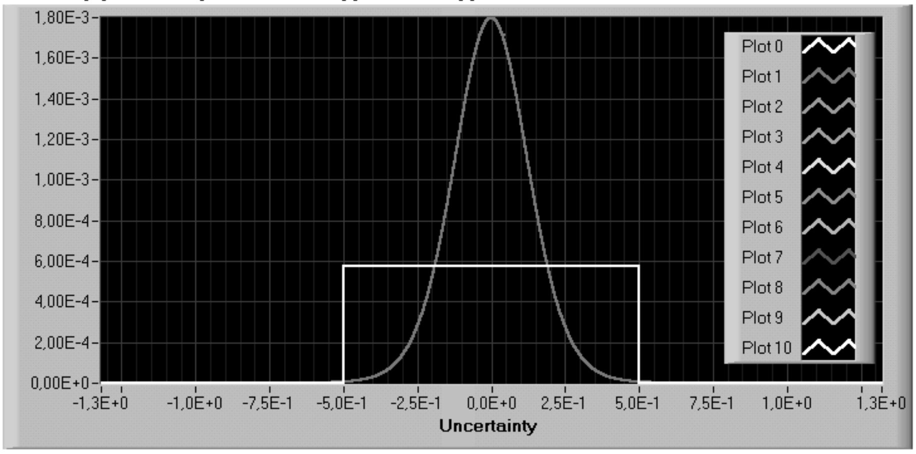
	c	sqrt(3)*uB	uA	n
1	1		0,126	16
2	1	0,5		
3	1			
4	1			
5	1			
6	1			
7	1			

Level of confidence, p

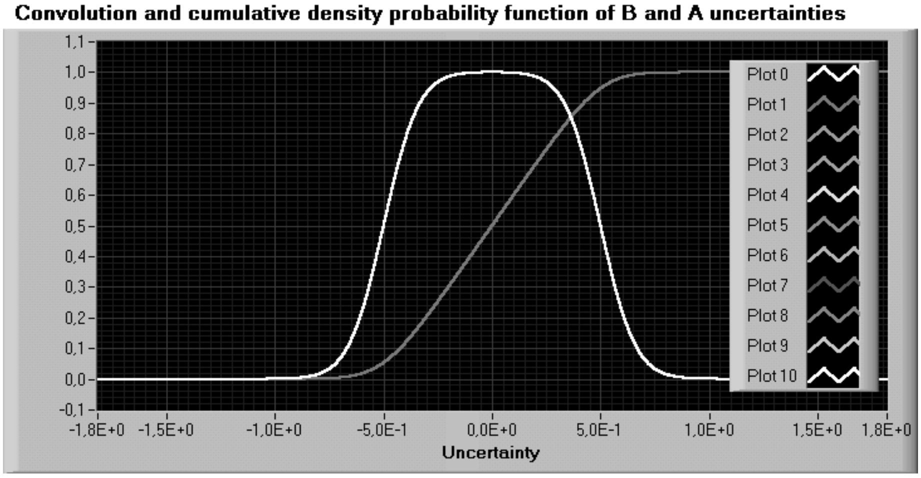
0,9900

Coverage factor, k	Expanded uncertainty, U
2,20122429	6,93330910E-1

Density probability function of type B and type A uncertainties



Convolution and cumulative density probability function of B and A uncertainties



Appendix 2 calculation for confidence level 0.95

The software requires: declaration of confidence level, all components of standard uncertainty of A and B type uncertainties, which are declared at www.metrol.p.lodz, after pressing enter, the users data are send to server, and after calculation values are returned back to users. Just to speed up transmission a screen of user can by a little different as shown below.

Table

	c	sqrt(3)*uB	uA	n
1	1		0,126	16
2	1	0,5		
3	1			
4	1			
5	1			
6	1			
7	1			

Level of confidence, p

0,9500

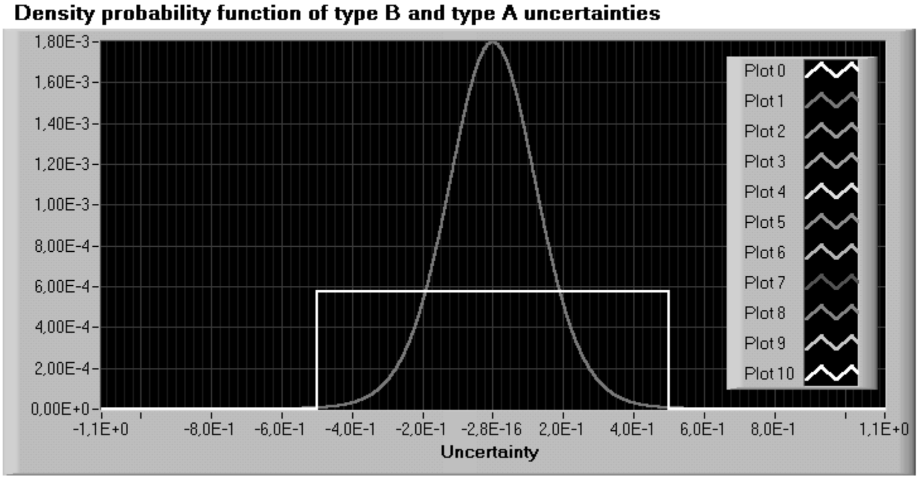
Coverage factor, k

1,81396863

Expanded uncertainty, U

5,71355007E-1

Density probability function of type B and type A uncertainties



Convolution and cumulative density probability function of B and A uncertainties